Math 20A. Lecture 12.

In this lecture we discuss the first part of the Fundamental Theorem of Calculus, which shows the ways in which differentiation and integration are inverse operations. We will discuss the second part in the next lecture.

Theorem 1 (The Fundamental Theorem of Calculus, Part 1)

Suppose that $y = F(x)$ is continuous and its derivative $r = F'(x)$ is piecewise continuous on an interval containing a and b. Then

$$\int_a^b F'(x)\,dx = F(b) - F(a). \tag{1}$$

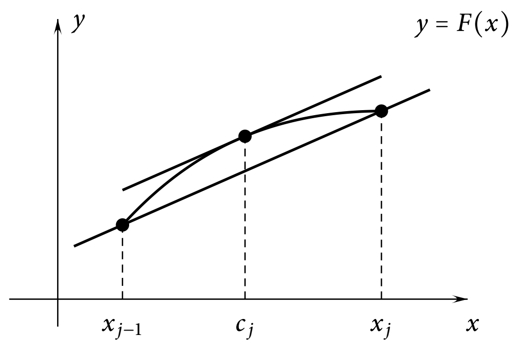
To show how this result can be proved in the case of $a < b$, we consider a partition

$$a = x_0 < x_1 < x_2 < \cdots < x_N = b$$

of $[a, b]$ into N subintervals whose endpoints include all points where $F'(x)$ is not defined. By the Mean Value Theorem on the interval $[x_{j-1}, x_j]$ (Figure 1), there is, for each j, a point c_j with $x_{j-1} < c_j < x_j$ such that

$$\frac{F(x_j) - F(x_{j-1})}{x_j - x_{j-1}} = F'(c_j). \tag{1}$$

FIGURE 1



We multiply both sides of equation (1) by $x_j - x_{j-1}$ and set $\Delta x_j = x_j - x_{j-1}$ to have for $j = 1, 2, \ldots, N$

$$F(x_j) - F(x_{j-1}) = F'(c_j)\Delta x_j.$$

This is the change in the value of F across the jth subinterval. The change in $F(x)$ from a to b is equal to sum of the changes in $F(x)$ across all N subintervals, so we obtain

$$\begin{aligned} F(b) - F(a) &= F(x_N) - F(x_0) \\ &= [F(x_1) - F(x_0)] + [F(x_2) - F(x_1)] + [F(x_3) - F(x_2)] + \cdots + [F(x_N) - F(x_{N-1})] \\ &= \sum_{j=1}^{N} [F(x_j) - F(x_{j-1})] = \sum_{j=1}^{N} F'(c_j)\Delta x_j. \end{aligned}$$

The sum on the right is a Riemann sum for $\int_a^b F'(x)\,dx$.

This, for each partition, there is at least one Riemann sum that equals $F(b) - F(a)$. Since the integral is the limit of all Riemann sums, the integral equals $F(b) - F(a)$, as is stated in the theorem.

It is convenient to write $\left[F(x)\right]_a^b$ for the difference $F(b) - F(a)$ in values of F. Then the conclusion of the Fundamental Theorem (1) reads

(2) $$\int_a^b F'(x)\,dx = \left[F(x)\right]_a^b.$$

Example 1 What is the value of $\int_0^2 \frac{d}{dx}(x^4)\,dx$?

Answer: Set $F(x) = x^4$, $a = 0$, and $b = 2$ in (2). •
$\int_0^2 \frac{d}{dx}(x^4)\,dx = \int_0^2 F'(x)\,dx = \left[F(x)\right]_0^2 = \left[x^4\right]_0^2 = [2^4] - [0^4] = 16$

Example 2 Figure 5 shows the graph of the continuous derivative $r = H'(x)$ of a continuous function $y = H(x)$. Region A in the drawing has area 89 and region B has area 62. What is $H(6) - H(1)$?

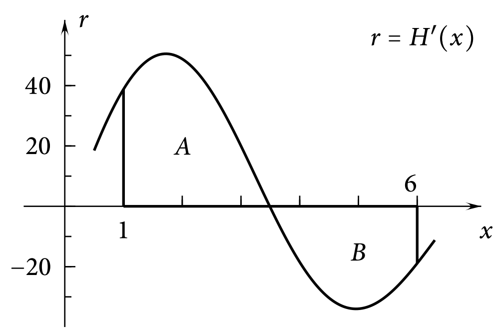


FIGURE 2

Answer: $H(6) - H(1) = \int_1^6 H'(x)\,dx$ = [Area A] − [Area B] = 89 − 62 = 27

Example 3 Figure 3 shows the rate of change $r = G'(t)$ of the United States gross national product (GNP) in three decades (Data from the Bureau of Economic Analysis). How much did the GNP change from from the beginning of 1970 to the beginning of 2000?

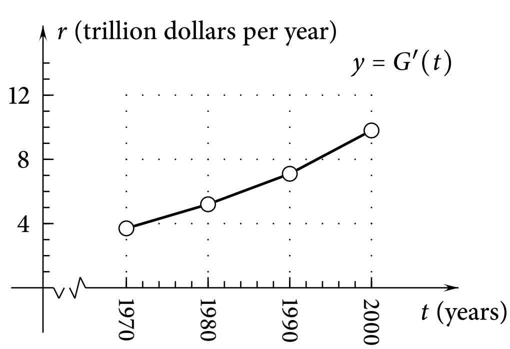


FIGURE 3

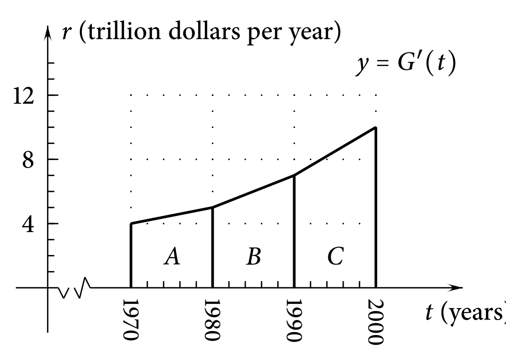


FIGURE 4

Answer: **(a)** $G(2000) - G(1970) = \int_{1970}^{2000} G'(t)\,dt$ • Figure 4 • The area of each trapezoid is equal to the average of the heights of its sides multipied by its width. •
$G(2000) - G(1970)$ = [Area of trapezoid A] + [Area of trapezoid B] + [Area of trapezoid C]
$\approx \frac{1}{2}(4 + 5)(10) + \frac{1}{2}(5 + 7)(10) + \frac{1}{2}(7 + 10)(10) = 45 + 60 + 85 = 190$ trillion dollars

Indefinite integrals

The symbol $\int f(x)\,dx$ is called an INDEFINITE INTEGRAL f. It represents an ANTIDERIVATIVE of the function f, a function whose derivatives is $f(x)$.

If we set $F'(x) = f(x)$ in the Fundamental Theorem (1), then $F(x) = \int f(x)\,dx$ and we obtain

$$\int_a^b f(x)\,dx = \left[\int f(x)\,dx\right]_a^b. \tag{3}$$

This equation describes the procedure we will use to evaluate a definite integral of a function f: We will find its indefinite integral and then use it in the Fundamental Theorem. Accordingly, we start by deriving integration formulas for indefinite integrals.

Integrals of $y = x^n$

Theorem 2 **(a)** *For any constant $n \neq -1$ and x in any open interval where x^n is defined,*

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C. \tag{4}$$

(b) *For x in $(-\infty, 0)$ or $(0, \infty)$,*

$$\int \frac{1}{x}\,dx = \ln|x| + C. \tag{5}$$

The C in formulas (4) and (5) represents an arbitrary constant. The term $+C$ is needed because if $F(x)$ is an antiderivative of f, then $F(x) + C$ is also an antiderivative for any choice of the constant C. This is because the constant has zero derivative.

Formula (4) for $n \neq -1$ follows from the calculation,

$$\frac{d}{dx}\left[\frac{1}{n+1}x^{n+1}\right] = \frac{n+1}{n+1}x^{n+1-1} = x^n.$$

Formula (5) for positive x is a consequence of the differentiation formula,

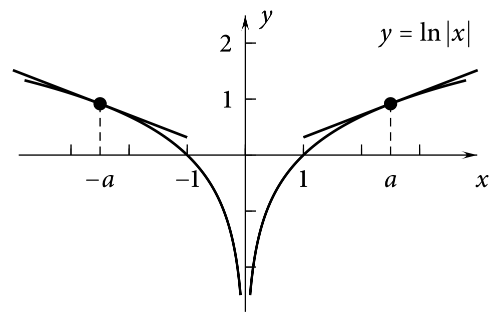
$$\frac{d}{dx}(\ln x) = \frac{1}{x}.$$

For $x < 0$, $|x| = -x$ and the Chain Rule gives

$$\frac{d}{dx}(\ln|x|) = \frac{d}{dx}(\ln(-x)) = \frac{1}{-x}\frac{d}{dx}(-x) = \frac{1}{x}$$

to establish (5) for $x < 0$.

The geometric interpretation of (5) for negative x is given by the graph of $y = \ln|x|$ in Figure 5. The portion for $x > 0$ is the graph of $\ln x$. The portion for $x < 0$ is the graph of $\ln(-x)$ and is the mirror image of the portion for $x > 0$. For positive a, the slope of the tangent line at $x = a$ has slope $1/a$ because the derivative of $\ln x$ is $1/x$. The tangent line at $x = -a$ has slope $1/(-a) = -1/a$ because it is the mirror image of the tangent line at $x = a$.



Slopes $\dfrac{1}{-a}$ and $\dfrac{1}{a}$
FIGURE 5

Example 4 Find the antiderivatives, $\int x^2\,dx$ of $y = x^2$.

Answer: $n = 2$ • $n + 1 = 3$ • $\dfrac{1}{n+1} = \frac{1}{3}$ • $\int x^2\,dx = \frac{1}{3}x^3 + C.$

We can check the answer to Example 4 by differentiation,

$$\frac{d}{dx}\left(\tfrac{1}{3}x^3 + C\right) = \tfrac{1}{3}(3x^2) = x^2.$$

Applying the Fundamental Theorem to Theorem 2 gives the corresponding result for definite integrals:

Theorem 3 **(a)** *For $n \neq -1$ and a and b in an open interval where x^n is defined,*

(6) $$\int_a^b x^n\,dx = \left[\frac{1}{n+1}x^{n+1}\right]_a^b = \frac{1}{n+1}b^{n+1} - \frac{1}{n+1}a^{n+1}.$$

(b) *For numbers a and b that are both positive or both negative,*

(7) $$\int_a^b \frac{1}{x}\,dx = \Big[\ln|x|\Big]_a^b = \ln|b| - \ln|a|.$$

Example 5 Evaluate $\int_1^4 x^{-2}\,dx$.

Answer: $n = -2.$ • $n + 1 = -1$ • $\dfrac{1}{n+1} = -1$ • $\int_1^4 x^{-2}\,dx = \Big[-x^{-1}\Big]_1^4 = \Big[\dfrac{-1}{x}\Big]_1^4 = -\frac{1}{4} - (-1) = \frac{3}{4}$

Example 6 What is the value of $\int_0^1 \left(4\sqrt[3]{x} + 6\sqrt{x}\right)dx$?

Answer: $\int_1^2 \left(4\sqrt[3]{x} + 6\sqrt{x}\right)dx = \int_1^2 (4x^{1/3} + 6x^{1/2})\,dx$ • For the first term, set $n = \frac{1}{3}$ for which $n + 1 = \frac{4}{3}$ and $\dfrac{1}{n+1} = \frac{3}{4}$ • For the second term, set $n = \frac{1}{2}$, for which $n + 1 = \frac{3}{2}$ and $\dfrac{1}{n+1} = \frac{2}{3}$ •
$\int_0^1 \left(4\sqrt[3]{x} + 6\sqrt{x}\right)dx = \int_0^1 (4x^{1/3} + 6x^{1/2})\,dx = \left[4(\tfrac{3}{4}x^{4/3}) + 6(\tfrac{2}{3}x^{3/2})\right]_0^1 = \left[3x^{4/3} + 4x^{3/2}\right]_0^1 =$
$[3(1^{4/3}) + 4(1^{3/2})] - [3(0^{4/3}) + 4(0^{3/2})] = 7$

Example 7 Find the area of the region bounded by the curve $y = 3x^2 - x^3$ and the x-axis in Figure 6. (Factor $3x^2 - x^3$ to determine where it is zero, positive, and negative.)

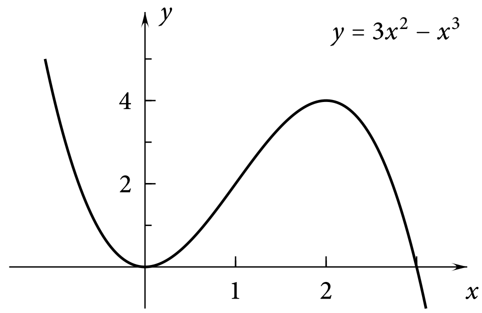


FIGURE 6

Answer: $3x^2 - x^3 = x^2(3-x)$ is zero at $x = 0$ and $x = 3$, is positive for $x < 0$ and for $0 < x < 3$, and is negative for $x > 3$. • $[\text{Area}] = \int_0^3 (3x^2 - x^3)\,dx$ • For $n = 2$: $n+1 = 3$ and $\frac{1}{n+1} = \frac{1}{3}$ • For $n = 3$: $n+1 = 4$ and $\frac{1}{n+1} = \frac{1}{4}$ • $[\text{Area}] = \int_0^3 (3x^2 - x^3)\,dx = \left[3(\tfrac{1}{3}x^3) - (\tfrac{1}{4}x^4)\right]_0^3 = \left[x^3 - \tfrac{1}{4}x^4\right]_0^3 = [3^3 - \tfrac{1}{4}(3^4)] - [0^3 - \tfrac{1}{4}(0^4)]$ $= [27 - \tfrac{81}{4}] - [0] = \tfrac{27}{4}$

Example 8 Suppose that the temperature in a room is 50°F at time $t = 0$ (hours) and that the rate of change of the temperature is $6\sqrt{t}$ degrees per hour for $0 \le t \le 4$. What is the temperature at $t = 4$?

Answer: One approach: Use a definite integral. • Let $T(t)$ denote the temperature at time t. • $T(0) = 50$ and $T'(t) = 6\sqrt{t}$ for $0 \le t \le 4$ • $T(4) = T(0) + \int_0^4 T'(t)\,dt = 50 + \int_0^4 6t^{1/2}\,dt$ $= 50 + \left[6(\tfrac{2}{3}t^{3/2})\right]_0^4 = 50 + \left[4t^{3/2}\right]_0^4 = 50 + [4(4^{3/2})] - [4(0^{3/2})] = 50 + 4(8) = 82°\text{F}$

(Calculations: $n = \frac{1}{2}$ • $n + 1 = \frac{3}{2}$ • $\frac{1}{n+1} = \frac{2}{3}$)

Another approach: Use an indefinite integral. • $T(t) = \int T'(t)\,dt = \int (6t^{1/2})\,dt = 6(\tfrac{2}{3}t^{3/2}) + C = 4t^{3/2} + C$ • Set $t = 0$: $T(0) = C$ and $T(0) = 50$ • $C = 50$ • $T(t) = 4t^{3/2} + 50$ • $T(2) = 4(2^{3/2}) + 50 = 4(8) + 50 = 82°\text{F}$

Example 9 **(a)** What is the area of the region bounded by the x-axis, the lines $x = 1$ and $x = 4$, and the curve $y = 1/x$? **(b)** What is the area of the region bounded by the x-axis, the lines $x = -4$ and $x = -1$, and the curve $y = 1/x$?

Answer: **(a)** $[\text{Area}] = \int_1^4 \frac{1}{x}\,dx = \left[\ln|x|\right]_1^4 = \ln|4| - \ln|1| = \ln(4)$

(b) The region is below the x-axis. • $[\text{Area}] = -\int_{-4}^{-1} \frac{1}{x}\,dx = \left[-\ln|x|\right]_{-4}^{-1} = (-\ln|-1|) - (-\ln|-4|) = \ln(4)$

Interactive Examples

Work the following Interactive Examples on the class web page, http//www.math.ucsd.edu/~ ashenk/
(The chapter and section numbers on this site do not match those in the textbook for the class.)

Section 6.3: 1-4

Section 6.5: 1–4