Math 20A. Lecture 13.

In this lecture we will derive integration formulas from the following differentiation formulas from earlier lectures:

(1) $$\frac{d}{dx}(e^x) = e^x \qquad \frac{d}{dx}(b^x) = (\ln b)b^x$$

(2) $$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

(3) $$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

(4) $$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

(5) $$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

Here are some suggestions for remembering these formulas:

The second equation in (1) can be obtained from the first by writing $b^x = e^{\ln(b^x)} = e^{x \ln b}$ and applying the Chain Rule:

$$\frac{d}{dx}(b^x) = \frac{d}{dx}(e^{x\ln b}) = e^{x\ln b}\frac{d}{dx}(x\ln b) = \ln(b)b^x.$$

The second equations in (2), (3), and (4) can be obtained from the first equations by interchanging the "functions", $\sin x$, $\tan x$ and $\sec x$, and the "cofunctions", $\cos x$, $\cot x$, and $\csc x$, and adding minus signs.

To avoid confusing the formulas in (5), remember that the inverse sine function $y = \sin^{-1} x$ is only defined for $-1 \le x \le 1$ and is derivative exists only for $-1 < x < 1$, while the inverse tangent function $y = \tan^{-1} x$ and its derivative are defined for all x.

Each of differentiation formulas (1) through (5) has a corresponding integration formula, given in the next theorem.

Theorem 1 *For $x$in the open integrals where the functions are defined,*

(6) $$\int e^x\, dx = e^x + C \qquad \int b^x\, dx = \frac{1}{\ln b}b^x + C$$

(7) $$\int \cos x\, dx = \sin x + C \qquad \int \sin x\, dx = -\cos x + C$$

(8) $$\int \sec^2 x\, dx = \tan x + C \qquad \int \csc^2 x\, dx = -\cot x + C$$

(9) $$\int \sec x \tan x\, dx = \sec x + C \qquad \int \csc x \cot x\, dx = -\csc x + C$$

(10) $$\int \frac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1} x + C \qquad \int \frac{1}{1+x^2}\, dx = \tan^{-1} x + C$$

The first formulas in (6) through (10) and the last formula in (10) are obtained by rephrasing the corresponding differentiation formulas in (1) through (5) as antidifferentiation formulas. The formula on the right of (6) is obtained by first dividing both sides of the formula on the right of (1) by $\ln b$. The formulas on the right of (7), (8), and (9) are obtained by first multiplying both sides of the equations on the right of (2), (3). and (4) by -1.

You do not need to memorize the formulas in Theorem 1. Recall them from the corresponding differentiation formulas when they are needed.

Example 1 **(a)** Find a formula for the function $y = g(x)$ such that $g'(x) = e^x$ for all x and $g(2) = 10$. **(b)** Check your answer.

Answer: (a) $g(x) = \int e^x\,dx = e^x + C$ • $g(2) = e^2 + 10$ and $g(2) = 10$ • $e^2 + 10 = C$ • $C = 10 - e^2$ • $g(x) = e^x + 10 - e^2$.

(b) Check: $g'(x) = \frac{d}{dx}(e^x + 10 - e^2) = e^x$ and $g(2) = e^2 + 10 - e^2 = 10$.

Example 2 **(a)** Perform the integration, $\int (2\sin x + 3\cos x)\,dx$. **(b)** Check your answer.

Answer: (a) $\int (2\sin x + 3\cos x)\,dx = 2\int \sin x\,dx + 3\int \cos x\,dx = -2\cos x + 3\sin x + C$.

(b) Check: $\frac{d}{dx}(-2\cos x + 3\sin x) = -2(-\sin x) + 3(\cos x) = 2\sin x + 3\cos x$.

Example 3 Perform the integration, $\int (\sec^2 x + \csc^2 x)\,dx$.

Answer: $\int (\sec^2 x + \csc^2 x)\,dx = \tan x - \cot x + C$

Each of the formulas for an indefinite integral in Theorem 1 can be used to evaluate definite integrals by applying the Fundamental Theorem in the form,

$$\int_a^b f(x)\,dx = \left[\int f(x)\,dx\right]_a^b.$$

Example 4 Evaluate $\int_0^1 \frac{1}{1+x^2}\,dx$.

Answer: $\int_0^1 \frac{1}{x^2+1}\,dx = \left[\int \frac{1}{x^2+1}\,dx\right]_0^1 = \left[\tan^{-1} x\right]_0^1 = \tan^{-1}(1) - \tan^{-1}(0) = \frac{1}{4}\pi - 0 = \frac{1}{4}\pi$

Example 5 Find the area of the region between the x-axis and the curve $y = x^2 + e^x$ for $-1 \le x \le 1$.

Answer: $y = x^2 + e^x$ is positive for $-1 \le x \le 1$ • $[\text{Area}] = \int_{-1}^1 (x^2 + e^x)\,dx = \left[\frac{1}{3}x^3 + e^x\right]_{-1}^1$
$= \left[\frac{1}{3}(1^3) + e^1\right] - \left[\frac{1}{3}(-1)^3 + e^{-1}\right] = [\frac{1}{3} + e] - [-\frac{1}{3} + e^{-1}] = \frac{2}{3} + e - e^{-1}$

Example 6 A car is 30 miles north of a town at time $t = 0$ (hours) and its velocity toward the north is $v(t) = 60 + 5\sin t$ miles per hour for $0 \le t \le 3$. Where is it at $t = 3$?

Answer: Use an indefinite integral. • Let $s = s(t)$ denote the car's distance north of the town at time t. • $s'(t) = v(t) = 60 + 5\sin t$ • $s(t) = \int (60 + 5\sin t)\,dt = 60t - 5\cos t + C$ • $s(0) = 30$ and $s(0) = -5 + C$ • $-5 + C = 30$ • $C = 35$ • $s(t) = 60t - 5\cos t + 35$ • $s(3) = 60(3) - 5\cos(3) + 35 = 215 - 5\cos(3)$ miles

Example 7 What is the area of the region between $y = \sec^2 x$ and the x-axis for $-1 \le x \le 1$?

Answer: $\sec^2 x > 0$ for $-1 \le x \le 1$ • [Area]
$= \int_{-1}^1 \sec^2 x\,dx = \left[\tan x\right]_{-1}^1 = \tan(1) - \tan(-1) = \tan(1) - [-\tan(1)] = 2\tan(1)$

Part 2 of The Fundamental Theorem

The first part of the Fundamental Theorem,

$$\int_a^b F'(x)\,dx = F(b) - F(a) \tag{11}$$

involves the integral of a derivative. Part 2 involves the derivative of an integral with a variable upper limit of integration.

Theorem 2 (The Fundamental Theorem, Part 2) *If f is continuous on an open interval I containing the point a, then the function $\int_a^x f(t)\,dt$ is differentiable on I and for all x in I,*

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x). \tag{12}$$

The variable t is used as the variable of integration in (2) because x is the upper limit of integration.

It is easy to derive Theorem 1 from Part 1 of the Fundamental Theorem (1) if we assume that f in Theorem 1 is the continuous derivative of a continuous function F on the interval being considered. Then for x in the interval,

$$\int_a^x f(t)\,dt = \int_a^x F'(t)\,dt = F(x) - F(a)$$

and therefore

$$\frac{d}{dx}\int_a^x f(t)\,dt = \frac{d}{dx}[F(x) - F(a)] = F'(x) = f(x).$$

This is formula (12). (The theorem, as stated, is important in the theory of calculus because it shows that every continuous function is a derivative of a continuous function.)

Example 8 Find the derivative $\dfrac{d}{dx}\displaystyle\int_1^x \sqrt{t^4+7}\,dt$.

Answer: $y = \sqrt{t^4+7}$ is continuous for all t. • $\dfrac{d}{dx}\displaystyle\int_1^x \sqrt{t^4+7}\,dt = \left[\sqrt{t^4+7}\right]_{t=x} = \sqrt{x^4+7}$.

Example 9 Find the derivative of $F(x) = \displaystyle\int_1^x t^3\,dt$ **(a)** by evaluating the integral and then finding the derivative and **(b)** by applying Theorem 1.

Answer: **(a)** $F(x) = \displaystyle\int_0^x t^3\,dt = \left[\tfrac{1}{4}t^4\right]_1^x = \tfrac{1}{4}x^4 - \tfrac{1}{4}$ • $F'(x) = \dfrac{d}{dx}(\tfrac{1}{4}x^4 - \tfrac{1}{4}) = x^3$

(b) By Theorem 1, $F'(x) = \dfrac{d}{dx}\displaystyle\int_0^x t^3\,dt = x^3$

Interactive Examples

Work the following Interactive Examples on the class web page, http//www.math.ucsd.edu/~ ashenk/ (The chapter and section numbers on this site do not match those in the textbook for the class.)

Section 6.4: 1

Section 6.7: 1–9