Math 20A, Lecture 2

One-sided and two-sided limits

If $y = f(x)$ is defined on an open interval (a, x_0) to the left of $x = x_0$ and the number $f(x)$ approaches a number L as x approaches x_0 from the left (Figure 1), we say that L is the LIMIT OF $f(x)$ AS x TENDS TO x_0 FROM THE LEFT, and write

$$\lim_{x \to x_0^-} f(x) = L \quad \text{or} \quad f(x) \to L \text{ as } x \to x_0^-.$$

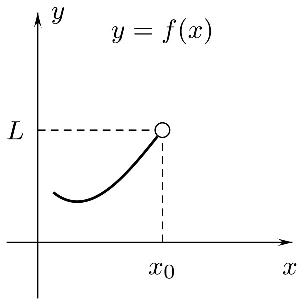


FIGURE 1

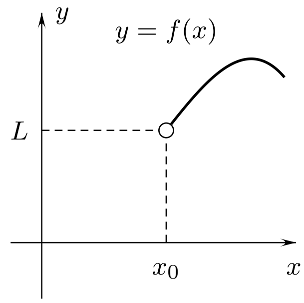


FIGURE 2

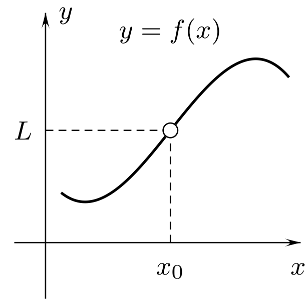


FIGURE 3

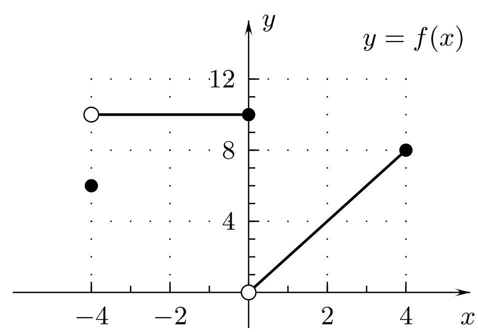
If $y = f(x)$ is defined on an open interval (x_0, b) to the right of $x = x_0$ and $f(x)$ approaches a number L as x approaches x_0 from the right (Figure 2), we say that L is the LIMIT OF $f(x)$ AS x TENDS TO x_0 FROM THE RIGHT, and write

$$\lim_{x \to x_0^+} f(x) = L \quad \text{or} \quad f(x) \to L \text{ as } x \to x_0^+.$$

If $y = f(x)$ is defined on an open interval (a, x_0) to the left of $x = x_0$ and on an open interval (x_0, b) to the right of x_0 and if $f(x)$ approaches a number L as x approaches x_0 from the left and from the right (Figure 3), we say that L is the LIMIT or the TWO-SIDED LIMIT of $f(x)$ as x tends to x_0. We write

$$\lim_{x \to x_0} f(x) = L \quad \text{or} \quad f(x) \to L \text{ as } x \to x_0.$$

Example 1 The graph of a function f defined for $-4 \le x \le 4$ is shown below. Find **(a)** $\lim_{x \to -4^+} f(x)$, **(b)** $\lim_{x \to 0^-} f(x)$, **(c)** $\lim_{x \to 0^+} f(x)$, **(d)** $\lim_{x \to 0} f(x)$, **(e)** $\lim_{x \to 2} f(x)$, and **(f)** $\lim_{x \to 4^-} f(x)$.



Answer: (a) $\lim_{x \to -4^+} f(x) = 10$ **(b)** $\lim_{x \to 0^-} f(x) = 10$ **(c)** $\lim_{x \to 0^+} f(x) = 0$ **(d)** $\lim_{x \to 0^+} f(x)$ does not exist. **(e)** $\lim_{x \to 2} f(x) = 4$ **(f)** $\lim_{x \to 4^-} f(x) = 8$

Theorem 1 If $\lim_{x\to x_0} f(x) = L$ and $\lim_{x\to x_0} g(x) = M$ with numbers L and M, then

$$\lim_{x\to x_0} [f(x) + g(x)] = L + M.$$

If also $M \neq 0$, then

$$\lim_{x\to x_0} \left[\frac{f(x)}{g(x)}\right] = \frac{L}{M}.$$

The two-sided limit in this theorem can be replaced by the one-sided limits as $x \to x_0^-$ or as $x \to x_0^+$.

Example 2 What is $\lim_{x\to 7} \dfrac{F(x)+G(x)}{F(x)-G(x)}$ if $\lim_{x\to 7} F(x) = 20$ and $\lim_{x\to 7} G(x) = 10$?

Answer: 3

Example 3 What is the limit of $P(x)Q(x)$ as $x \to 5^-$ if $P(x) \to 3$ and $Q(x) \to 7$ as $x \to 5^-$?

Answer: 21

Theorem 2 Suppose that f is a constant function, a power function, an exponential function, a logarithm, a trigonometric function, or an inverse trigonometric function, or is given by a single formula constructed from such functions with sums, differences, products, quotients, or compositions. Then for any point x_0 in the interior of the domain of f,

$$\lim_{x\to x_0} f(x) = f(x_0).$$

Example 4 Find $\lim_{x\to 2} \sqrt{1+x^2}$.

Answer: $\sqrt{5}$

Example 5 What is $\lim_{x\to 0} \left(\dfrac{\cos x}{e^x+1}\right)$?

Answer: $\lim_{x\to 0} \left(\dfrac{\cos x}{e^x+1}\right) = \frac{1}{2}$

Example 6 **(a)** Sketch the graph of the function P defined below. Then find **(b)** $P(-1)$, **(c)** $\lim_{x\to -1^-} P(x)$, **(d)** $\lim_{x\to -1^+} P(x)$, **(e)** $\lim_{x\to -1} P(x)$, and **(f)** $\lim_{x\to 0} P(x)$.

$$P(x) = \begin{cases} -3/x & \text{for} \quad x < -1 \\ 1 & \text{for} \quad -1 \le x \le 1 \\ 3/x & \text{for} \quad x > 1. \end{cases}$$

Answer: **(a)** Figure A6 **(b)** $P(-1) = 1$ **(c)** $\lim_{x\to -1^-} P(x) = 3$ **(d)** $\lim_{x\to -1^+} P(x) = 1$ **(e)** $\lim_{x\to -1} P(x)$ does not exist. **(f)** $\lim_{x\to 0} P(x) = 1$

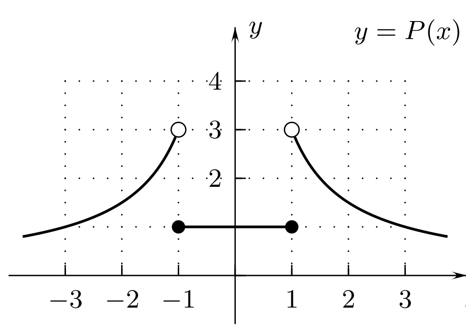


Figure A6

Example 7 Find **(a)** $f(10)$, **(b)** $\lim\limits_{x\to 10^-} f(x)$, **(c)** $\lim\limits_{x\to 10^+} f(x)$, and **(d)** $\lim\limits_{x\to 2} f(x)$ where

$$f(x) = \begin{cases} x^2 & \text{for} \quad x < 10 \\ 5 & \text{for} \quad x = 10 \\ x^{-2} & \text{for} \quad x > 10. \end{cases}$$

Answer: (a) $f(10) = 5$ **(b)** $\lim\limits_{x\to 10^-} f(x) = 100$ **(c)** $\lim\limits_{x\to 10^+} f(x) = \frac{1}{100}$ **(d)** $\lim\limits_{x\to 10} f(x)$ is not defined.

Example 8 Find $\lim\limits_{x\to -2} \dfrac{x^2-4}{x+2}$ by rewriting the expression for the function so it does not have a denominator that tends to zero as $x \to -2$.

Answer: For $x \neq -2$, $\dfrac{x^2-4}{x+2} = x - 2$ • $\lim\limits_{x\to -2} \dfrac{x^2-4}{x+2} = -4$

Example 9 Find $\lim\limits_{x\to 2} \dfrac{1-2/x}{x-2}$ by rewriting the formula for the function so it does not have a denominator that tends to zero as $x \to 2$.

Answer: (a) For $x \neq 2$: $\dfrac{1-2/x}{x-2} = \dfrac{1}{x}$ • $\lim\limits_{x\to 2} \dfrac{1-2/x}{x-2} = \frac{1}{2}$

Example 10 Find $\lim\limits_{x\to 4} \dfrac{\sqrt{x}-2}{x-4}$ by making the substitution $z = \sqrt{x}$.

Answer: $z = \sqrt{x} \to \sqrt{4} = 2$ as $x \to 4$ • $x = z^2$ • $\dfrac{\sqrt{x}-2}{x-4} = \dfrac{z-2}{z^2-4} = \dfrac{1}{z+2} \to \frac{1}{4}$ as $z \to 2$

Continuity at a point

A function $y = f(x)$ is CONTINUOUS FROM THE LEFT at $x = x_0$ (Figure 4) if $\lim\limits_{x\to x_0^-} f(x) = f(x_0)$. The function is CONTINUOUS FROM THE RIGHT at $x = x_0$ (Figure 5) if $\lim\limits_{x\to x_0^-} f(x) = f(x_0)$. The function is CONTINUOUS at $x = x_0$ (Figure 6) if $\lim\limits_{x\to x_0} f(x) = f(x_0)$.

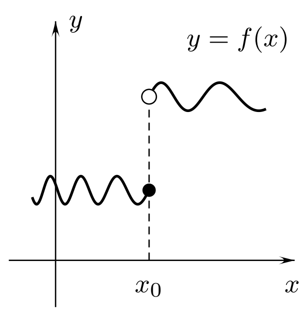


FIGURE 4

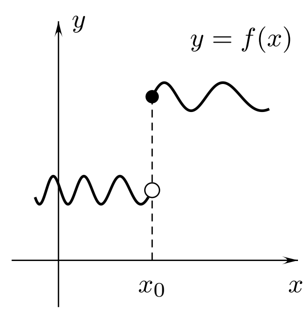


FIGURE 5

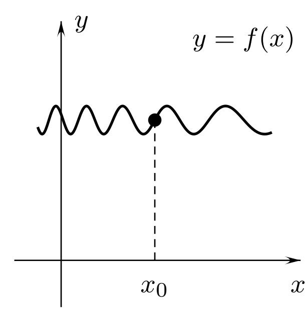
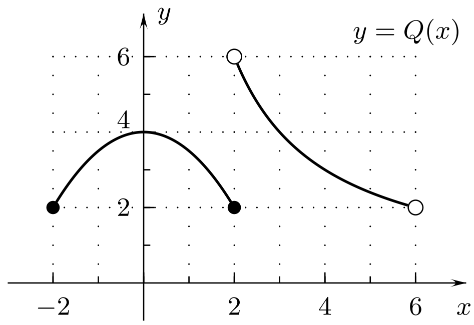


FIGURE 6

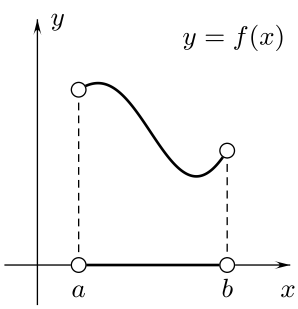
Example 11 **(a)** The graph of a function Q defined for $-2 \le x \le 6$ is shown below. At what values of x is Q continuous? **(b)** At what points is Q continuous from the right but not continuous? **(c)** At what points is Q continuous from the left but not continuous?



Answer: (a) Q is continuous at all x_0 for $-2 < x_0 < 2$ and for $2 < x_0 < 6$ **(b)** Q is continuous from the right at $x = -2$ but not continuous there. **(c)** Q is continuous from the left at $x = 2$ but not continuous there.

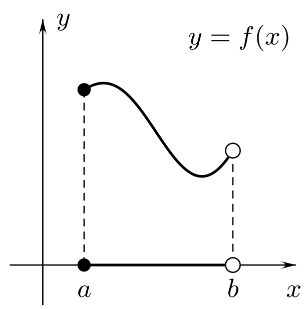
Continuity on intervals

A function $y = f(x)$ is CONTINUOUS ON AN INTERVAL if it continuous at all points in the interior of the interval, is continuous from the right at the left endpoint of the interval if it has one, and is continuous from the left at the right endpoint of the interval if it has one. Figures 7 through 10 illustrate this definition.



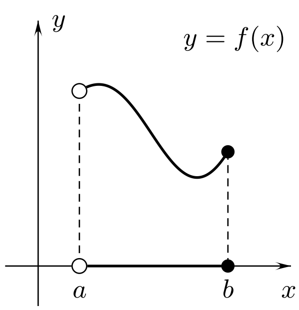
f is continuous on (a, b).

FIGURE 7



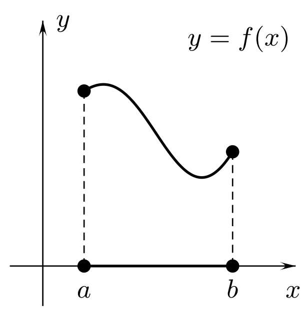
f is continuous on $[a, b)$.

FIGURE 8



f is continuous on $(a, b]$.

FIGURE 9



f is continuous on $[a, b]$.

FIGURE 10

Example 12 What are the largest intervals on which the function M of Figure 11 is continuous?

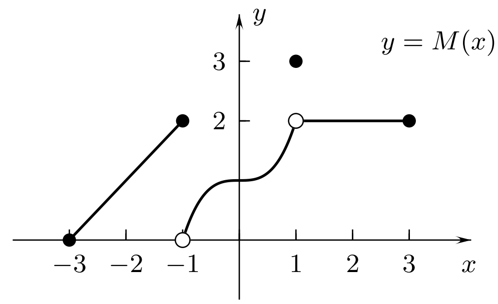


FIGURE 11

Answer: $[-3,-1]$, $(-1,1)$, and $(1,3]$

Theorem 3 Suppose that f is a constant function,a power function, an exponential function, a logarithm, a trigonometric function, or an inverse trigonometric function, or is given by a single formula constructed from such functions with sums, differences, products, quotients, or compositions. Then f is continuous on all intervals in its domain.

Example 13 **(a)** What are the largest intervals on which $g(x) = \dfrac{\cos x}{\sqrt{x}-1}$ is continuous? (In dealing with infinite intervals, we say that an interval I_1 is "larger" than another interval I_2 if I_2 is contained in I_1.)
(b) Find $\lim\limits_{x\to 0^+} g(x)$.

Answer: (a) The largest intervals on which g is continuous are $[0,1)$ and $(1,\infty)$. **(b)** $\lim\limits_{x\to 0^+} g(x) = -1$

Example 14 **(a)** Sketch the graph of the function h defined below. **(b)** What are the largest intervals on which it is continuous?

$$h(x) = \begin{cases} 2/x & \text{for} \quad x < -1 \\ 2x^3 & \text{for} \quad -1 \le x \le 1 \\ 2/x & \text{for} \quad x > 1. \end{cases}$$

Answer: (a) Figure A14 **(b)** h is continuous on $(-\infty,\infty)$.

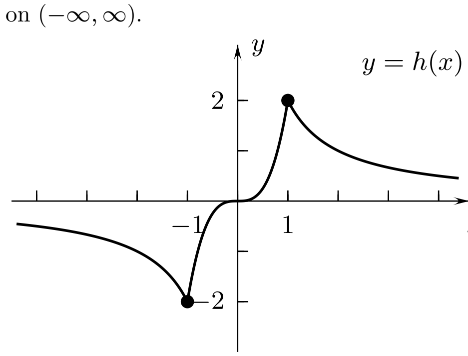
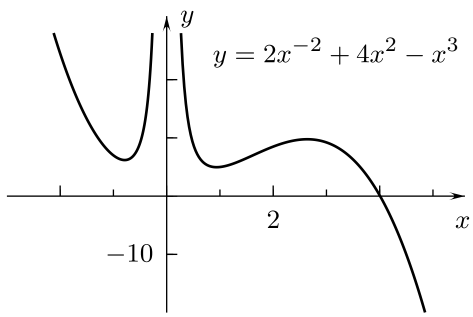


Figure A14

Infinite limits

We say that a variable TENDS TO ∞ if it becomes an arbitrarily large positive number and that it TENDS TO $-\infty$ if it becomes an arbitrarily large negative number. (We say that a negative number x is "large" if $|x|$ is large.)

Limits where the variable or the value of a function tends to ∞ or to $-\infty$ are illustrated in Figures 12 and 13.

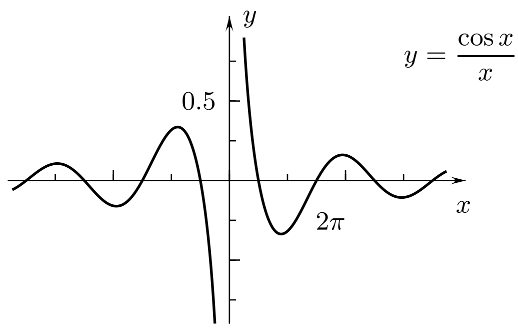


$y \to \infty$ as $x \to -\infty$

$y \to \infty$ as $x \to 0$

$y \to -\infty$ as $x \to \infty$

FIGURE 12



$y \to 0$ as $x \to -\infty$

$y \to -\infty$ as $x \to 0^-$

$y \to \infty$ as $x \to 0^+$

$y \to 0$ as $x \to \infty$

FIGURE 13

Interactive Examples on the class web page

Section 1.1: 1–4

Section 1.2: 1–5

Section 1.3: 1–5