Math 20A. Lecture 3.

Linear functions and constant rates of change

A function is called LINEAR if its graph is a line. Rates of change of these functions can be found without calculus.

Example 1 **(a)** A van is $s = 200 + 60t$ miles east of a truck stop at time t (hours). Draw the graph of this function in a ts-plane **(b)** Because the graph is a line, the van's velocity is constant. What is the velocity and how does it relate to the graph?

Answer: **(a)** The line has slope 60 and s-intercept 200. • Figure 1 **(b)** Figure 2 • The van's velocity between times $t = a$ and $t = b$ with $a \neq b$ is $\dfrac{\Delta s}{\Delta t} = \dfrac{60(b-a)}{b-a} = 60$ miles per hour. • The velocity is the slope of the graph.

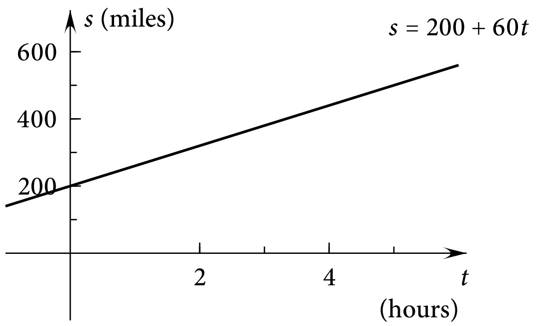


FIGURE 1

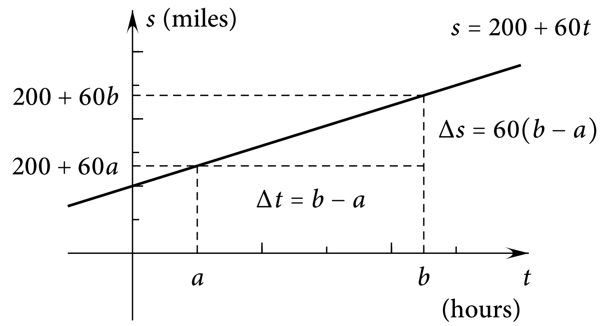


FIGURE 2

The velocity in Example 1 is the RATE OF CHANGE of the van's position $s = 200 + 60t$ with respect to the time t. This example illustrates a general principle:

Theorem 1 *The (constant) rate of change of a linear function $y = f(x)$ with respect to its variable x is the slope of its graph.*

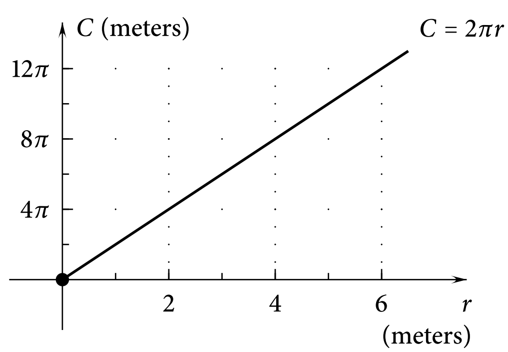
Example 2 What is the rate of change of $f(x) = 7 - 6x$ with respect to x?

Answer: -6

Example 3 The circumference C of a circle of radius r meters is $C = 2\pi r$ (meters). What is the rate of change of the circumference with respect to the radius?

Answer: The rate of change of C with respect to r is the slope 2π (meters per meter) of the line $C = 2\pi r$ (Figure 3).

FIGURE 3



Average rates of change

Definition 1 *The* AVERAGE RATE OF CHANGE *of $y = f(x)$ with respect to x from $x = a$ to $x = b$ for $a \neq b$ is*

(1)
$$\frac{[\text{Change in } f(x)]}{[\text{Change in } x]} = \frac{f(b) - f(a)}{b - a}.$$

It is equal to the slope of the secant line through the points at $x = a$ and $x = b$ on the graph of f (Figure 4).

$$\begin{bmatrix}\text{Average rate}\\ \text{of change}\end{bmatrix} = \frac{\text{Rise}}{\text{Run}} = \frac{f(b) - f(a)}{b - a}$$

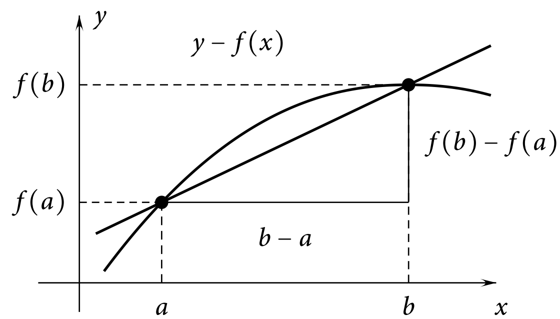


FIGURE 4

Example 5 What is the average rate of change of $f(x) = x + e^x$ with respect to x from $x = 0$ to $x = 3$?

Answer: $[\text{Average rate of change}] = \dfrac{(3 + e^3) - (0 + e^0)}{3 - 0} = \frac{1}{3}(2 + e^3)$

Average rate of change of position with respect to time is average velocity.

Example 6 An object is at $s = 50 - 50/t$ (yards) on an s-axis at time t (minutes) for $t \geq 1$. **(a)** Find its average velocity for $1 \leq t \leq 5$. **(b)** Sketch the graph of $s = s(t)$ and draw the secant line whose slope is the average velocity.

Answer: **(a)** 10 yards per minute **(b)** Figure 5

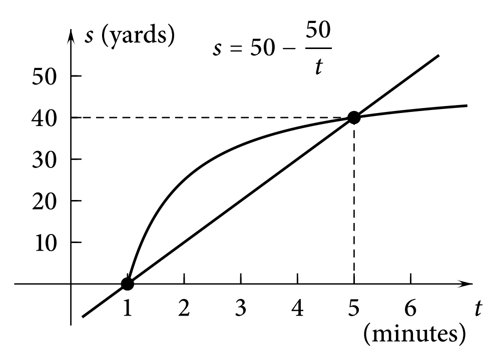
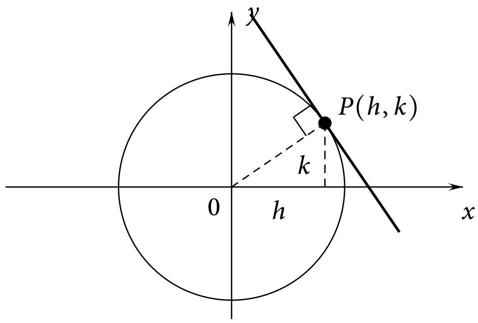


FIGURE 5

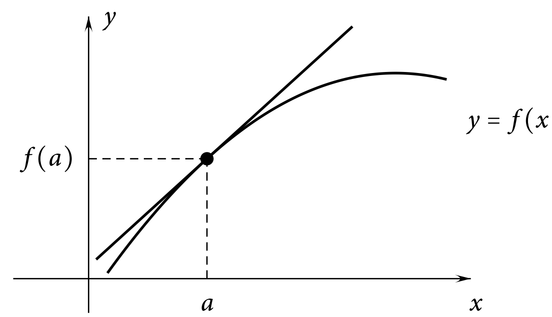
Tangent lines and derivatives

Euclid (c. 300 BC) defined a tangent line to a circle at a point P to be the line that intersects the circle at only that point (Figure 6). Then he proved that the tangent line is perpendicular to the radius at that point . This property can be used to find the slope of the tangent line: Because the radius OP in Figure 6 has slope k/h, the perpendicular tangent line at P has slope $-h/k$.



Tangent line at P

FIGURE 6



Tangent line at $x = a$

FIGURE 7

The graph $y = f(x)$ in Figure 7 does not have a similar geometric property that could be used to find the slope of its tangent line shown in the drawing. Instead, we find the tangent line as a limit of secant lines. As we saw above, the slope of the secant line through the points at $x = a$ and $x = b$ for $b \neq a$ (Figure 8) equals the rise $f(b) - f(a)$ from the point at $x = a$ to the point at $x = b$, divided by the corresponding run $b - a$:

(2) $$[\text{Slope of the secant line}] = \frac{f(b) - f(a)}{b - a}.$$

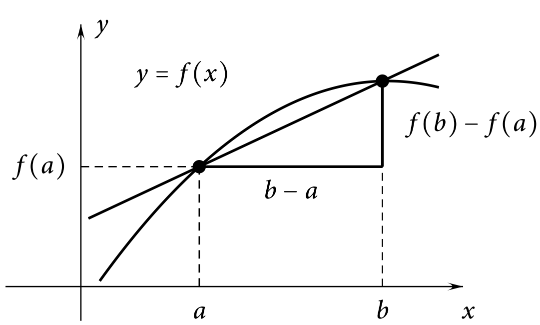


FIGURE 8

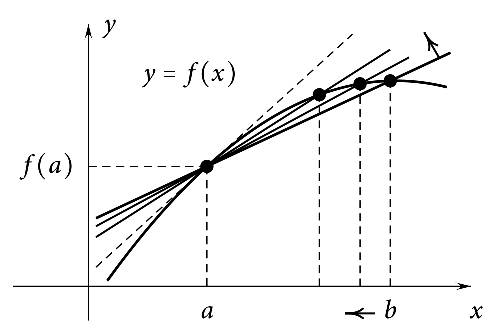


FIGURE 9

Imagine what happens in Figure 8 as the number b approaches a. The secant line turns about the fixed point at $x = a$ (Figure 9). If its slope (2) has a finite limit, then the secant line approaches the line with that slope—the dashed diagonal line in Figure 9—and this line is the tangent line to the graph of f at $x = a$ that is shown in Figure 7. This leads to the following definition.

Definition 2 *The* DERIVATIVE *of $y = f(x)$ at $x = a$ is*

$$f'(a) = \lim_{b \to a} \frac{f(b) - f(a)}{b - a} \tag{3}$$

provided this limit exists and is finite. The derivative $f'(a)$, if it exists, is the SLOPE OF THE TANGENT LINE *to the graph of f at $x = a$ and is the* RATE OF CHANGE *or* INSTANTANEOUS RATE OF CHANGE *of f with respect to x at a.*

The derivative is called the rate of change because it is the limit of the slopes of secant lines, which are average rates of change. The process of finding derivatives is called DIFFERENTIATION.

The average rate of change (1) is called a DIFFERENCE QUOTIENT because it is a quotient of the differences $f(b) - f(a)$ and $b - a$. With this terminology there are three terms for average rate of change and three terms for (instantaneous) rate of change, which are related as shown the following diagram, where the vertical arrows indicate that b is approaching a:

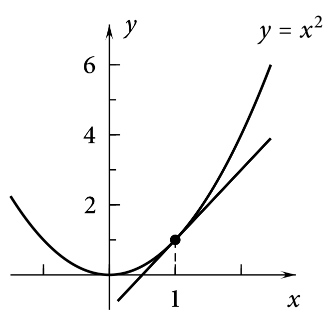
$\frac{f(b) - f(a)}{b - a}$	=	Average rate of change	=	Slope of the secant line	=	The difference quotient
↓		↓		↓		↓
$f'(a)$	=	Instantaneous rate of change	=	Slope of the tangent line	=	The derivative

INSTANTANEOUS VELOCITY is the instantaneous rate of change with respect to time of an object's coordinate $s = s(t)$.

Predicting a derivative by calculating difference quotients

Figure 10 shows the graph of $f(x) = x^2$ and its tangent line at $x = 1$, whose slope equals the derivative $f'(1)$.

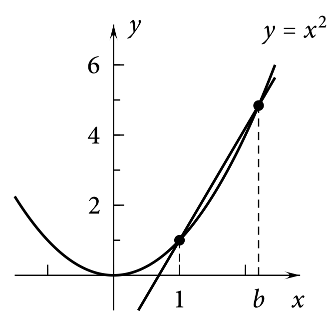
We can predict the value of this derivative by calculating the slopes of the secant line $\dfrac{f(b) - f(1)}{b - 1}$ (Figure 11) for values of b close to 1.



Tangent line

Slope = $f'(1)$

FIGURE 10



Secant line

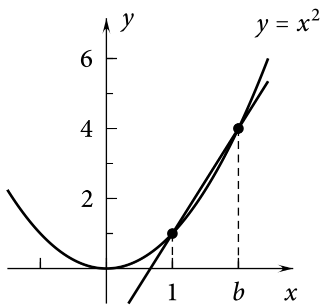
Slope = $\dfrac{f(b) - f(1)}{b - 1}$

FIGURE 11

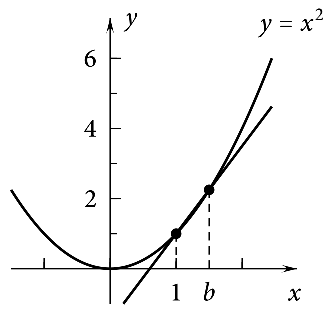
Example *7* Predict the derivative $f'(1)$ for $f(x) = x^2$ by calculating the average rate of change of f with respect to x from $x = 1$ to $x = b$ for $b = 2$, $b = 1.5$, $b = 1.25$, $b = 1.1$, $b = 1.001$, and $b = 1.00001$.

Answer: The average rates of change in the table below suggest that the derivative $f'(1) = \lim_{b \to 1} \dfrac{b^2 - 1}{b - 1}$ is 2. The secant lines corresponding to the values in the table are shown in Figures 12 through 17.

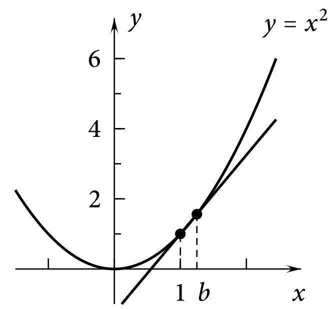
b	2	1.5	1.25	1.1	1.001	1.00001
$\dfrac{b^2-1}{b-1} =$	3	2.5	2.25	2.1	2.001	2.00001



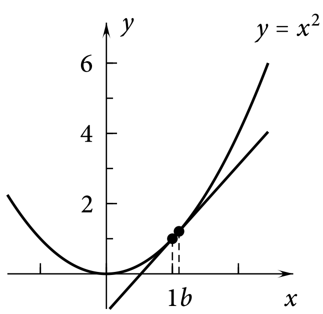
$b = 2$
FIGURE 12



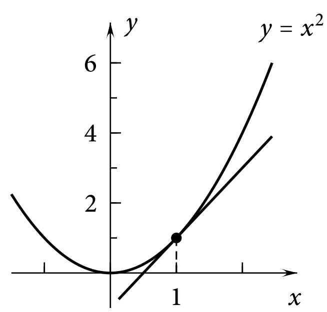
$b = 1.5$
FIGURE 13



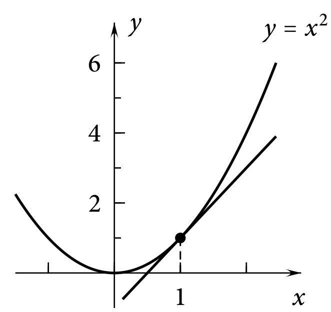
$b = 1.25$
FIGURE 14



$b = 1.1$
FIGURE 15



$b = 1.001$
FIGURE 16



$b = 1.00001$
FIGURE 17

Finding exact derivatives

To make the algebra in finding exact derivatives easier to follow, we use x instead of b for the x-coordinate of the variable point in the definition. Then the secant line passes through $(a, f(a))$ and $(x, f(x))$, as shown in Figure 18, and the definition takes the form

(4)
$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Secant line of slope

$$\frac{f(x) - f(a)}{x - a}$$

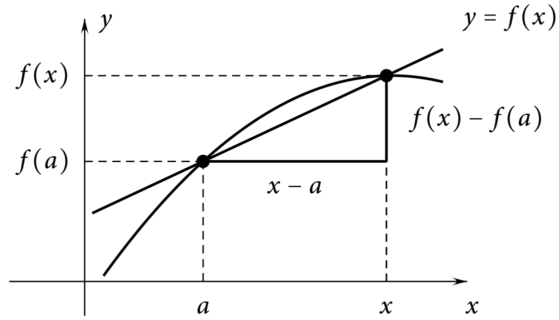


FIGURE 18

Example 8 Use Definition (4) to find the derivative of $f(x) = x^2$ at $x = 1$.

Answer: $f'(1) = \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} (x + 1) = \Big[x + 1\Big]_{x=1} = 2$

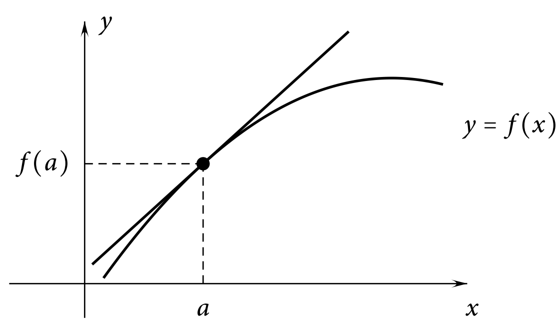
Example 9 Use Definition 4 to find $g'(0)$ for $g(x) = 3x - x^3$.

Answer: $g'(0) = \lim_{x \to 0} \frac{3x - x^3}{x} = \lim_{x \to 0} (3 - x^2) = \Big[3 - x^2\Big]_{x=0} = 3 - 0^2 = 3$

Equations of tangent lines

The tangent line to $y = f(x)$ at $x = a$ in Figure 19 passes through the point $(a, f(a))$ and has slope $f'(a)$. By the point-slope formula, it has the equation,

(5)
$$y = f(a) + f'(a)(x - a).$$



$y = f(a) + f'(a)(x - a)$

FIGURE 19

Example 10 Give an equation of the tangent line to the graph of $f(x) = x^2$ at $x = 1$ and draw it with the graph of the function.

Answer: $f'(1) = 2$ by Example 8 • Tangent line: $y = 1 + 2(x - 1)$ or $y = 2x - 1$ • Figure 20

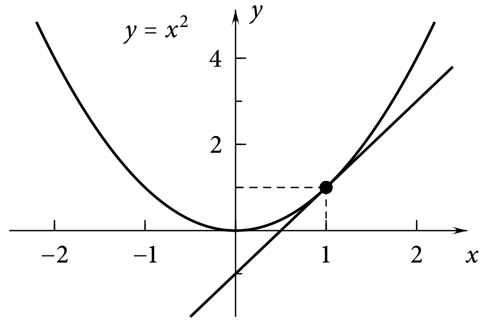


FIGURE 20

The Δx-definition of the derivative

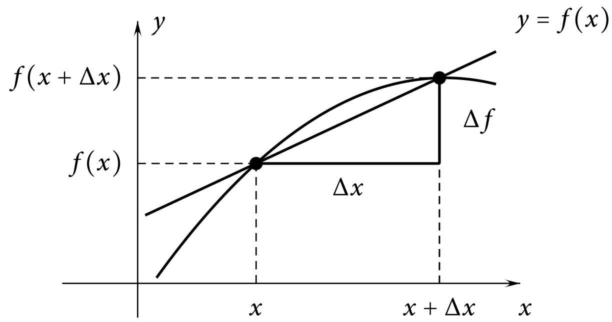
It is often convenient to modify the definition of the derivative by having x be the fixed value where we want to find the derivative and denote the x-coordinate of the variable point by $x + \Delta x$, where Δ is the capital Greek letter delta. Then Δx is the run from $(x, f(x))$ to $(x + \Delta x, f(x + \Delta x))$ on the secant line, as shown in Figure 21. We denote the corresponding rise $f(x + \Delta x) - f(x)$ by Δf. With this notation, the definition of the derivative becomes

(6) $$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}.$$

Secant line of slope

$$\frac{\Delta f}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

FIGURE 21



Example 10 Use formulation (6) of the definition to find the derivative of $f(x) = 1/x$ at an arbitrary $x \neq 0$.

Answer: $f'(x) = \lim_{\Delta x \to 0} \dfrac{\dfrac{1}{x + \Delta x} - \dfrac{1}{x}}{\Delta x}$ • For $\Delta x \neq 0$,

$$\frac{\dfrac{1}{x + \Delta x} - \dfrac{1}{x}}{\Delta x} = \frac{1}{\Delta x}\left[\frac{1}{x + \Delta x} - \frac{1}{x}\right] = \frac{1}{\Delta x}\left[\frac{x - (x + \Delta x)}{x(x + \Delta x)}\right] = \frac{1}{\Delta x}\left[\frac{-\Delta x}{x(x + \Delta x)}\right] = \frac{-1}{x(x + \Delta x)}$$ •

$$f'(x) = \lim_{\Delta x \to 0} \frac{-1}{x(x + \Delta x)} = \frac{-1}{x^2}$$

Finding an approximate rate of change from a graph

The function $A = A(t)$ whose graph is shown in Figure 22 gives the percentage of alcohol in a person's blood t hours after he or she has consumed three fluid ounces of alcohol. (Data adapted from *Encyclopædia Britannica*) As you can see from the graph, the blood-alcohol level rises from 0% to about 0.22% in about two hours, and then drops close to 0.01% after ten more hours.

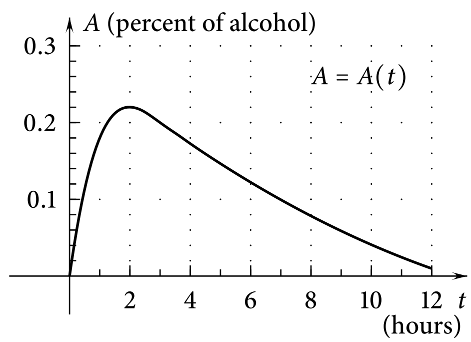


FIGURE 22

Example 11 Find the approximate rates of change with respect to time of the blood-alcohol level in Figure 22

(a) after one hour and **(b)** after eight hours.

Answer: **(a)** The approximate tangent line at $t = 1$ in Figure 23 passes through the points $P = (1, 0.18)$ and $Q = (2, 0.28)$. • $A'(1) \approx = \dfrac{(0.28 - 0.18) \text{ percent}}{(2-1) \text{ hour}} = 0.1$ percent per hour. **(b)** The approximate tangent line in Figure 24 passes through the points $R = (4, 0.16)$ and $S = (8, 0.08)$. •

$A'(8) \approx \dfrac{(0.08 - 0.16) \text{ percent}}{(8-4) \text{ hours}} = -0.02$ percent per hour.

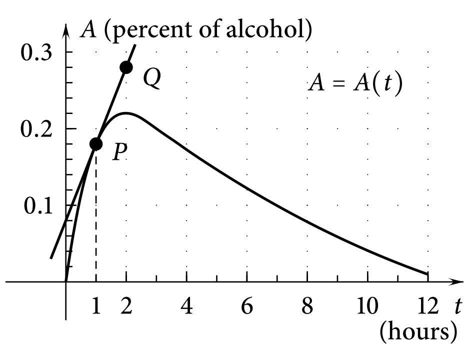


FIGURE 23

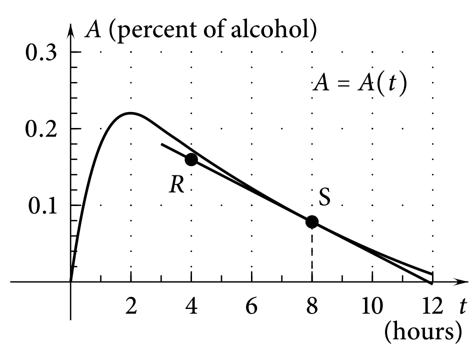


FIGURE 24

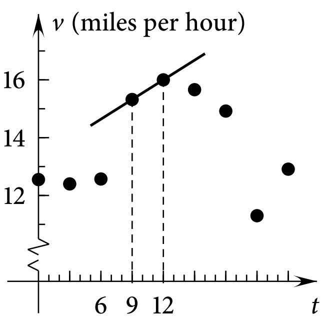
Finding an approximate rate of change from a table

Example 12 The next table lists the wind speed at three-hour intervals one day in Dodge City, Kansas. (Data from *Wind Energy Systems*, Prentice Hall International, Inc.) The time t is measured in hours with $t = 0$ at midnight. Use this data to estimate the rate of change $v'(t)$ of the wind's velocity with respect to time at 9:00 AM ($t = 9$).

t	0	3	6	9	12	15	18	21	24
v(t)	12.55	12.40	12.57	15.32	16.00	15.66	14.92	11.30	12.91

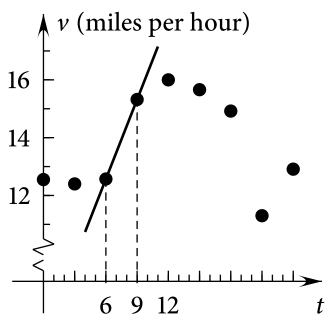
Answer: One solution: Use the secant line in Figure 25, whose slope is the average rate of change of $v = v(t)$ with respect to t for $9 \le t \le 12$ (a RIGHT-DIFFERENCE QUOTIENT). •

$v'(9) \approx \dfrac{v(12) - v(9)}{12 - 9} = \dfrac{16.00 - 15.32}{3} = \frac{1}{3}(0.68) \doteq 0.23$ miles per hour per hour



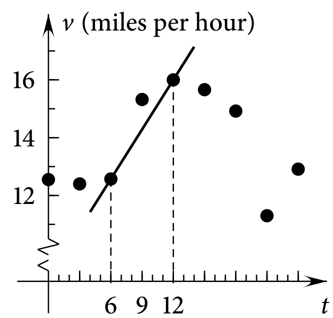
The slope is a right-difference quotient.

FIGURE 25



The slope is a left-difference quotient.

FIGURE 26



The slope is a centered difference quotient.

FIGURE 27

A second solution: Use the secant line in Figure 26, whose slope is the average rate of change for $6 \le t \le 9$ (a LEFT-DIFFERENCE QUOTIENT). • $v'(9) \approx \dfrac{v(9) - v(6)}{9 - 6} = \dfrac{15.32 - 12.57}{3} = \frac{1}{3}(2.75) \doteq 0.92$ miles per hour per hour

A third solution: Use the secant line in Figure 27, whose slope is the average rate of change for $6 \le t \le 12$ (a CENTERED DIFFERENCE QUOTIENT). • $v'(9) \approx \dfrac{v(12) - v(6)}{12 - 6} = \dfrac{16.00 - 12.57}{6} = \frac{1}{6}(3.43) \doteq 0.57$ miles per hour per hour

Exercises for Homework 1

Exercise 1 The length L of a copper rod is a linear function of the temperature T°F. The rod is 100 inches long at 50°F and expands 0.093 inches for every degree is temperature is increased. Give a formula for L as a function of T.

Answer: $L = 100 + 0.093(T - 50)$

Exercise 2 Figure 28 shows the graph of a linear function $O = O(V)$ that is a fairly good model of a human's oxygen absorption, measured in liters per minute, as a function of the rate of his or her breathing, also measured in liters per minute. Based on this mathematical model, what is the rate of change of oxygen absorption with respect to the rate of breathing?

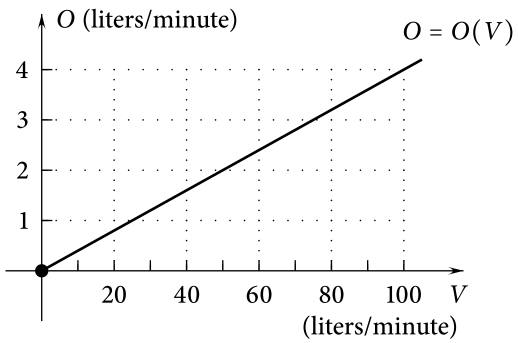


FIGURE 28

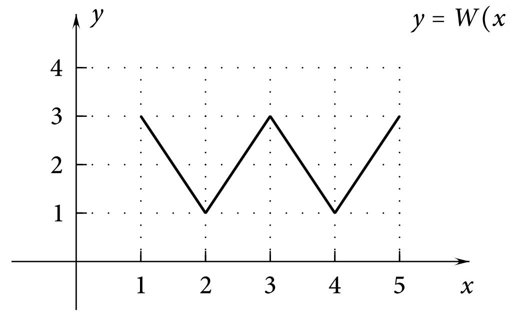


FIGURE 29

Answer: $\frac{1}{25}$ liters per minute per liters per minute

Exercise 3 The graph of a function W is shown in Figure 29. Find the average rates of change of W with respect to x **(a)** for $2 \le x \le 5$, **(b)** for $1 \le x \le 3$, and **(c)** for $2.50 \le x \le 2.51$.

Answer: **(a)** [Average rate of change for $2 \le x \le 5$] $\frac{2}{3}$ **(b)** [Average rate of change for $1 \le x \le 3$] = 0
(c) [Average rate of change for $2.50 \le x \le 2.51$] = 2

Exercise 4 Calculate the slope of the secant line through the points at $x = a$ and $x = b$ on the graph of $f(x) = 1/x^2$ with $a = 1$ and $b = 0.5, 0.75, 0.9, 0.999$ and 0.99999, and use the results to predict the value of the derivative $f'(1)$.

Answer: Prediction: $f'(1) = -2$

Exercise 5 The following table gives the velocity of sound in water $v = v(T)$ (feet per second) as a function of the temperature T°F of the water. (Data from the *CRC Handbook of Chemistry and Physics*.) **(a)** Does the velocity of sound increase or decrease as the temperature increases? **(b)** Based on this data, does the velocity decrease more rapidly when the temperature is 210°F or when it is 410°F?

T°F	200	240	280	320	360	400	440
$v(T)$ (feet per second)	5079	5000	4879	4724	4537	4331	4081

Answer: **(a)** $v(T)$ decreases as T increases. **(b)** The velocity decreases more rapidly at 410°F.

Exercise 6 Figure 30 shows the graph of a bird's height $h = h(t)$ above the ground as a function of time. **(a)** Approximately how fast is the bird's height decreasing at $t = 2$? **(b)** Approximately how fast is the bird's height increasing at $t = 6$?

Answer: **(a)** One answer: $h'(2) \approx \dfrac{31-55}{4-2} = -12$ feet per minute at $t = 2$ **(b)** One answer: $h'(6) \approx \dfrac{35-22}{8-6} = 6.5$ feet per minute

Exercise 7 Figure 31 is the graph of the force $F = F(s)$ exerted by a monkey ligament as a function of how much it is stretched. (Data adapted from *Human Adaptation and Accommodation*. The University of Michigan Press.) The ligament can easily be stretched very small amounts. This gives the portion of the graph for small s the "toe" shape shown in the figure. The portion of the curve that looks like a line is the "linear region," and the rest of the curve on the right is the "fatigue region", where the ligament weakens because increasing numbers of microfibers in it are torn. What is the approximate rate of change of the force with respect to the length in the linear region?

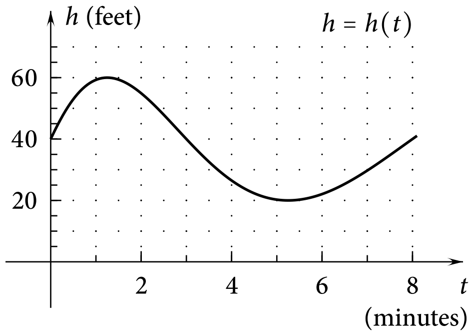


FIGURE 30

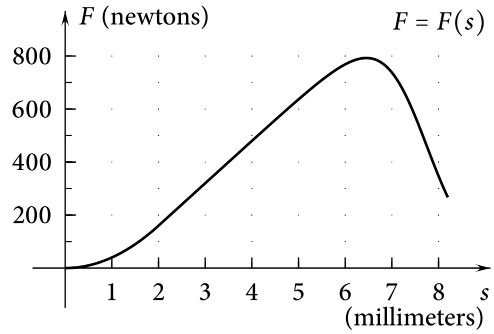


FIGURE 31

Answer: The rate of change in the linear region is approximately $F'(4) \approx \dfrac{800 - 480}{6 - 4} = 160$ newtons per millimeter.

Exercise 8 Vertebrae in the human spine are separated by fibrous elastic disks that compress when the load on them increases. The load on the disk is called the STRESS, and the percent of compression is called the STRAIN. The function $S = S(L)$ in Figure 32 gives the strain S of lower back (lumbar) vertebral disks as a function of the stress L on them. (Data adapted from *Physics with Illustrative Examples from Medicine and Biology*, Reading, Massachusetts: Addison-Wesley) Sketch the graph of the derivative $r = S'(L)$.

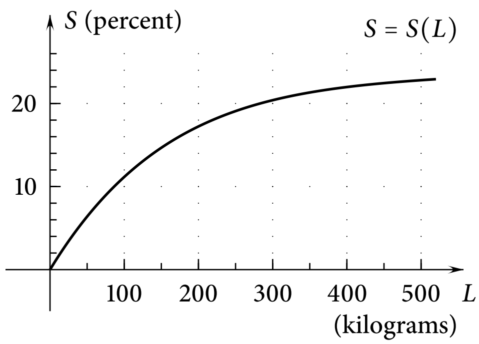


FIGURE 32

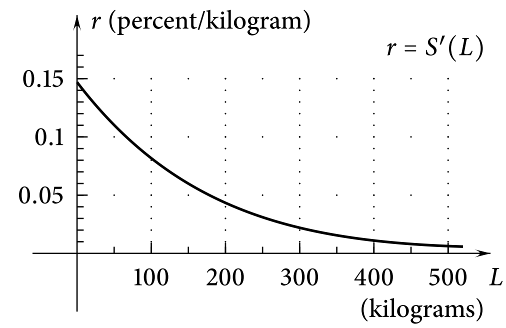


FIGURE 33

Answer: One answer: Figure 33

Interactive Examples

Work the following Interactive Examples on the class web page, http//www.math.ucsd.edu/~ ashenk/ (The chapter and section numbers on this site do not match the chapters and sections of the textbook.)

Section 2.1: 1–6

Section 2.2: 1–4

Section 2.5: 1–5