Math 20A. Lecture 4.

Leibniz notation and the differentiation operator

Isaac Newton (1642–1727) and Gottfried Leibniz (1646–1716), who are considered to be the founders of calculus, each introduced notation for the derivative. PRIME NOTATION $f'(a)$ is similar to that used by Newton, and the symbols used by Leibniz evolved into what is known today as LEIBNIZ NOTATION.

In Leibniz notation, the derivative f' of f is denoted $\dfrac{df}{dx}$ and its value $f'(a)$ at $x = a$ is denoted $\left[\dfrac{df}{dx}\right]_{x=a}$, so that the definition of the derivative reads

$$\left[\frac{df}{dx}\right]_{x=a} = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}. \tag{1}$$

The symbol $\dfrac{d}{dx}$ is called the DIFFERENTIATION OPERATOR. It converts a function f into its derivative:

$$\frac{d}{dx}(f) = \frac{df}{dx}.$$

The derivative of $y = x^n$ with constant n

Leibniz notation is used in the statement of the next theorem.

Theorem 1 *For any constant n and for x in the open interval or intervals where $y = x^{n-1}$ is defined,*

$$\frac{d}{dx}(x^n) = nx^{n-1} \tag{2}$$

In the case of $n = 0$, equation (2) is interpreted as the statement,

$$\frac{d}{dx}(1) = 0$$

which is valid for all x.

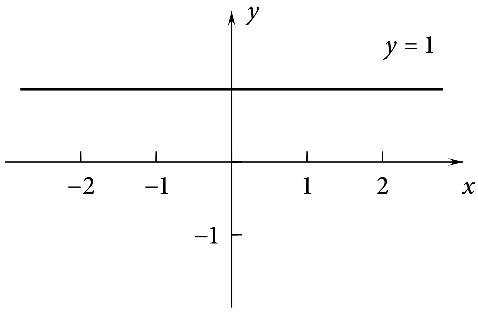
The derivative (2) exists for all x with two exceptions. It does not exist for $x \le 0$ if n is irrational or if n is a fraction p/q with p an odd integer and q an even integer, since then the domain of $y = x^n$ does not include any negative numbers; and it does not exist at $x = 0$ if n is < 1 since then $y = x^{n-1}$ is not defined at $x = 0$.

The cases of $n = 0$ and $n = 1$

For $n = 0$, the function $y = x^n$ is $y = 1$ since $x^0 = 1$, and for $n = 1$ the function $y = x^n$ is $y = x$ since $x^1 = x$. Because these functions are linear. their derivatives are the slopes of the lines that are their graphs (Figures 1 and 2). Consequently,

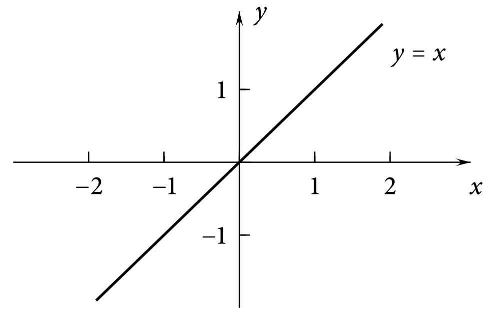
$$\frac{d}{dx}(x^0) = \frac{d}{dx}(1) = 0 \quad \text{and} \quad \frac{d}{dx}(x^1) = \frac{d}{dx}(x) = 1.$$

These formulas give (2) for $n = 0$ and $n = 1$.



$$\frac{d}{dx}(1) = [\text{Slope}] = 0$$

FIGURE 1



$$\frac{d}{dx}(x) = [\text{Slope}] = 1$$

FIGURE 2

The case of integers $n \geq 2$

To find the derivative of $y = x^n$ at any point $x = a$ for an integer $n \geq 2$, we use the definition,

$$\left[\frac{d}{dx}(x^n)\right]_{x=a} = \lim_{x \to a} \frac{x^n - a^n}{x - a}. \tag{3}$$

We found this limit for $n = 2$ and specific values of a in examples and exercises of the last lecture by using the factorization,

$$x^2 - a^2 = (x - a)(x + a).$$

Since $x + a \to a + a = 2a$ as $x \to a$, we conclude from the last equation that

$$x^2 - a^2 = (x - a)[\text{An expression that tends to } 2a \text{ as } x \to a].$$

There is a similar factorization for every integer $n \geq 3$. The first few read

$$\begin{aligned}
x^3 - a^3 &= (x - a)(x^2 + ax + a^2) \\
&= (x - a)[\text{An expression that tends to } 3a^2 \text{ as } x \to a] \\
x^4 - a^4 &= (x - a)(x^3 + ax^2 + a^2x + a^3) \\
& (x - a)[\text{An expression that tends to } 4a^3 \text{ as } x \to a] \\
x^5 - a^5 &= (x - a)(x^4 + ax^3 + a^2x^2 + ax^4 + x^4) \\
&= (x - a)[\text{An expression that tends to } 5a^4 \text{ as } x \to a]
\end{aligned}$$

This pattern continues, so that for any integer $n \geq 2$,

$$x^n - a^n = (x - a)[\text{An expression that tends to } na^{n-1} \text{ as } x \to a].$$

Substituting this formula into definition (3) gives

$$\begin{aligned}
\left[\frac{d}{dx}(x^n)\right]_{x=a} &= \lim_{x \to a} \frac{x^n - a^n}{x - a} \\
&= \lim_{x \to a} \frac{(x - a)[\text{An expression that tends to } na^{n-1} \text{ as } x \to a]}{x - a}.
\end{aligned}$$

We cancel the factor $(x - a)$ to obtain

$$\left[\frac{d}{dx}(x^n)\right]_{x=a} = \lim_{x \to a} [\text{An expression that tends to } na^{n-1} \text{ as } x \to a] = na^{n-1}.$$

This gives Theorem 1 in this case at $x = a$.

***Example** 1* What is the derivative of $y = x^6$?

Answer: If $n = 6$, then $n - 1 = 5$ • Apply Theorem 1 with $n = 6$. • $\frac{d}{dx}(x^6) = 6x^5$

The case of positive fractions $n = p/q$

To show more clearly the ideas behind the proof of Theorem 1 for fractions $n = p/q$, where p and q are positive integers, we will only deal with the case of $n = \frac{4}{3}$.

By definition (1),

$$\left[\frac{d}{dx}(x^{4/3})\right]_{x=a} = \lim_{x \to a} \frac{x^{4/3} - a^{4/3}}{x - a}.$$

For $x \neq a$ we set $z = x^{1/3}$ and $b = a^{1/3}$. Then $x = z^3$, $a = b^3$ and

$$\frac{x^{4/3} - a^{4/3}}{x - a} = \frac{z^4 - b^4}{z^3 - b^3}.$$

Dividing the numerator and denominator of the last ratio by $z - b$ yields

$$\frac{x^{4/3} - a^{4/3}}{x - a} = \frac{\dfrac{z^4 - b^4}{z - b}}{\dfrac{z^3 - b^3}{z - b}}. \tag{4}$$

The derivative of $x^{4/3}$ is the limit of this expression as x tend to a. When we take this limit, $z = x^{1/3}$ tends to $b = a^{1/3}$ so that the numerator of (4) tends to the derivative $4b^3$of $y = z^4$ at $z = b$ and the denominator tends to the derivative $3b^2$ of $y = z^3$ at $x = b$.

$$\left[\frac{d}{dx}(x^{4/3})\right]_{x=a} = \frac{\left[\dfrac{d}{dz}(z^4)\right]_{z=b}}{\left[\dfrac{d}{dz}(z^3)\right]_{z=b}} = \frac{4b^3}{3b^2} = \frac{4}{3}b.$$

Since $b = a^{1/3}$, we obtain,

$$\left[\frac{d}{dx}(x^{4/3})\right]_{x=a} = \frac{4}{3}a^{1/3}.$$

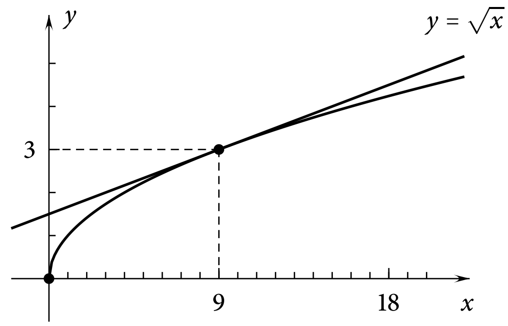
This is Theorem 1 at $x = a$ for $n = \frac{4}{3}$ for which $n - 1 = \frac{1}{3}$. Similar calculations would give the result for any positive fraction $n = p/q$.

***Example** 2* Give an equation for the tangent line to $y = \sqrt{x}$ at $x = 9$ and draw it with the curve.

Answer: $y(9) = \sqrt{9} = 3$ • $y'(9) = \left[\frac{d}{dx}(x^{1/2})\right]_{x=9} = \left[\frac{1}{2}x^{-1/2}\right]_{x=9} = \left[\frac{1}{2\sqrt{x}}\right]_{x=9} = \frac{1}{6}$ •

Tangent line: $y = y(9) + y'(9)(x - 9)$ or $y = 3 + \frac{1}{6}(x - 9)$ • Figure 3

FIGURE 3



The case of negative rational numbers n

We give the details only in the particular case of $n = -\frac{1}{3}$ to avoid confusing details.

By the definition,

$$\begin{aligned}\left[\frac{d}{dx}(x^{-1/3})\right]_{x=a} &= \lim_{x\to a}\frac{x^{-1/3}-a^{-1/3}}{x-a}\\ &= \lim_{x\to a}\frac{1}{x-a}\left(\frac{1}{x^{1/3}}-\frac{1}{a^{1/3}}\right)\\ &= \lim_{x\to a}\frac{1}{x-a}\left(\frac{a^{1/3}-x^{1/3}}{a^{1/3}x^{1/3}}\right)\\ &= \left(\lim_{x\to a}\frac{-1}{a^{1/3}x^{1/3}}\right)\left(\lim_{x\to a}\frac{x^{1/3}-a^{1/3}}{x-a}\right)\\ &= \frac{-1}{a^{1/3}a^{1/3}}\left[\frac{d}{dx}(x^{1/3})\right]_{x=a}.\end{aligned}$$

By Theorem 1 for positive fractions n, $\frac{d}{dx}(x^{1/3}) = \frac{1}{3}x^{-2/3}$ and the last equations give

$$\left[\frac{d}{dx}(x^{-1/3})\right]_{x=a} = \frac{-1}{a^{2/3}}\left(\tfrac{1}{3}a^{-2/3}\right) = -\tfrac{1}{3}a^{-4/3}.$$

This gives Theorem 1 at $x = a$ in the case of $n = -\frac{1}{3}$ for which $n - 1 = -\frac{4}{3}$.

We will establish Theorem 1 for irrational exponents n in a later lecture by using logarithms.

Derivatives of linear combinations of functions

A LINEAR COMBINATION of functions $y = f(x)$ and $y = g(x)$ is a function of the form $y = Af(x) + Bg(x)$ with constants A and B. Our next differention rule enables us to find the derivative of any such function if we know the derivatives of f and g.

Theorem 2 *If $y = f(x)$ and $y = g(x)$ have derivatives at x, then for any constants A and B*

(5) $$\frac{d}{dx}[Af(x) + Bg(x)] = Af'(x) + Bg'(x).$$

Theorem 2 is also established by using the definition of the derivative. Suppose that f and g have derivatives at $x = a$. Then

$$\begin{aligned}\left[\frac{d}{dx}[Af(x)+Bg(x)]\right]_{x=a} &= \lim_{x\to a}\frac{[Af(x)+Bg(x)]-[Af(a)+Bg(a)]}{x-a}\\ &= A\left[\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\right]+B\left[\lim_{x\to a}\frac{g(x)-g(a)}{x-a}\right]\\ &= Af'(a)+Bg'(a).\end{aligned}$$

Example 3 What is the derivative of $h(x) = 3f(x) + 5g(x)$ at $x = 10$ if $f'(10) = 4$ and $g'(10) = -2$?

Answer: $h'(10) = 3f'(10) + 5g'(10) = 3(4) + 5(-2) = 2$

Example 4 Find a formula for the derivative of $y = 5x^3 - x^{-2} + 4$.

Answer: $\frac{d}{dx}(5x^3 - x^{-2} + 4) = 15x^2 + 2x^{-3}$

Derivatives of functions involving roots of x and reciprocals of powers and roots are found by converting to exponential notation:

Example 5 What is $f'(x)$ for $f(x) = 7\sqrt[3]{x} + \dfrac{8}{\sqrt{x}}$?

Answer: $f'(x) = 7\frac{d}{dx}(x^{1/3}) + 8\frac{d}{dx}(x^{-1/2}) = 7\left(\frac{1}{3}x^{(1/3)-1}\right) + 8\left(-\frac{1}{2}x^{-(1/2)-1}\right) = \frac{7}{3}x^{-2/3} - 4x^{-3/2}$

Example 6 A man driving on a straight road is $s = \frac{1}{4}t^3 + 50t + 40$ miles from his home t hours after noon. What is his car's velocity at 4:00 PM?

Answer: [Velocity at time t] $= s'(t) = \frac{d}{dt}\left(\frac{1}{4}t^3 + 50t + 40\right) = \frac{3}{4}t^2 + 50$ • [Velocity at 4:00 PM] $= s'(4) = \left[\frac{3}{4}t^2 + 50\right]_{t=4} = \frac{3}{4}(4^2) + 50 = 62$ miles per hour

Example 7 A precision heater is controlled by varying the current supplied to it. It produces $Q(I) = 100I^2$ Calories of heat in one second when the current is I amperes. **(a)** What is the (instantaneous) rate of change of Q with respect to I at $I = 3$? **(b)** Give an equation for the tangent line to the graph $Q = Q(I)$ at $I = 3$ **(c)** Draw the graph and the tangent line in an IQ-plane.

Answer: **(a)** $Q'(I) = \frac{d}{dI}(100I^2) = 200I$ • $Q'(3) = 200(3) = 600$ Calories per ampere

(b) $Q(3) = 100(3^2) = 900$ • Tangent line: $Q = Q(3) + Q'(3)(I - 3)$ • $Q = 900 + 600(I - 3)$

(c) The points $(0,0)$ and $(3,100)$ are on the curve, which curves up to the right from the origin. • The points $(3,100)$ and $(5,2100)$ are on the tangent line. • Figure 4

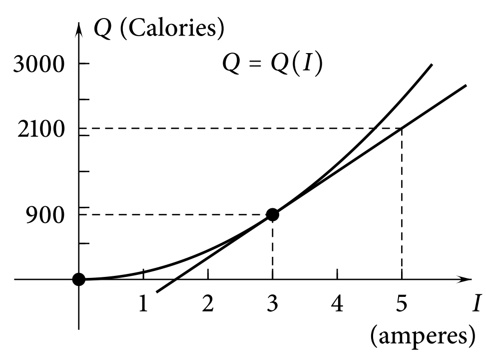


FIGURE 4

Differentiable functions are continuous

We will need the following fundamental result to derive the next differentiation rules.

Theorem 3 *If $y = f(x)$ has a derivative at $x = a$, then it is continuous at $x = a$.*

This theorem follows easily from the definitions. Suppose that the derivative,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(x)}{x - a}$$

exists. We write for $x \neq a$,

$$f(x) - f(a) = \left[\frac{f(x) - f(x)}{x - a}\right](x - a)$$

This quantity tends to zero as $x \to a$ because the difference quotient in square brackets tends to the number $f'(a)$ and $x - a$ tends to zero. Consequently, $f(x) \to f(a)$ as $x \to a$ and f is continuous at $x = a$.

The Product Rule

Imagine that the sides of the rectangle in Figure 5 are changing, so that the width $w = w(t)$, height $h = h(t)$, and area $A(t) = w(t)h(t)$ of the rectangle are functions of the time t. We consider a fixed time t and assume that $w'(t)$ and $h'(t)$ exist. To simplify the geometric interpretation of this discussion, we also assume that these derivatives are positive.

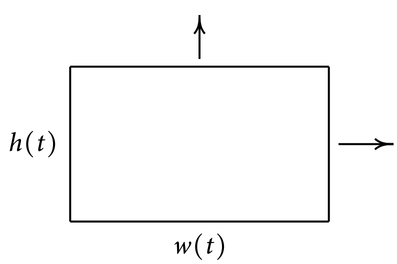


FIGURE 5

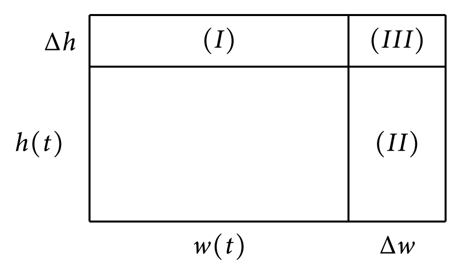


FIGURE 6

Consider a positive change Δt in the time from t to $t + \Delta t$. We let Δw and Δh be the corresponding changes in the width and height, as in Figure 6. Then at time $t + \Delta t$, the width of the rectangle is $w + \Delta w$ and its height is $h + \Delta h$. The change ΔA in the area from t to $t + \Delta t$ is the area of the three rectangles labeled (I), (II), and (III) in Figure 6:

$$\Delta A = [\text{Area } (I)] + [\text{Area } (II)] + [\text{Area } (III)]$$

Rectangle (I) is $w(t)$ units wide and Δh units high, rectangle (II) is Δw units wide and $h(t)$ units high, and rectangle (III) is Δw units wide and Δh high. Therefore,

$$\Delta A = w(t)\Delta h + h(t)\Delta w + \Delta w \Delta h. \tag{6}$$

This equation also holds for negative Δt. We divide both sides of it by Δt to have

$$\frac{\Delta A}{\Delta t} = w(t)\left[\frac{\Delta h}{\Delta t}\right] + h(t)\left[\frac{\Delta w}{\Delta t}\right] + \Delta w\left[\frac{\Delta h}{\Delta t}\right]. \tag{7}$$

The value of t is fixed, so $w(t)$ and $h(t)$ are constants. Also, since $h'(t)$ exists, h is continuous at t and

$$\lim_{\Delta t \to 0} \Delta h = 0.$$

Since $\dfrac{\Delta w}{\Delta t} \to w'(t)$ and $\dfrac{\Delta h}{\Delta t} \to h'(t)$ as $\Delta t \to 0$, equation (7) gives

$$\begin{aligned} A'(t) &= \lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = w(t)h'(t) + h(t)w'(t) + 0[h'(t)] \\ &= w(t)h'(t) + h(t)w'(t). \end{aligned} \tag{8}$$

Formula (8) is an example of the Product Rule. For its general statement, we replace t by x, w by f, h by g, and the area A by fg:

Theorem 4 (The Product Rule) *If $y = f(x)$ and $y = g(x)$ have derivatives at x, then so does their product, $y = f(x)g(x)$, and*

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}. \tag{9}$$

In prime notation (9) reads

$$(fg)' = fg' + gf'.$$

Remember the Product Rule as the following statement: The derivative of a product of two functions equals the first function multiplied by the derivative of the second, plus the second function multiplied by the derivative of the first.

Example 8 Find the derivative of $y = (x^5 + x^2)(x^{1/3} + 1)$ at $x = 1$.

Answer: $y'(x) = \dfrac{d}{dx}[(x^5 + x^2)(x^{1/3} + 1)] = (x^5 + x^2)\dfrac{d}{dx}(x^{1/3} + 1) + (x^{1/3} + 1)\dfrac{d}{dx}(x^5 + x^2) =$ $(x^5 + x^2)(\frac{1}{3}x^{-2/3}) + (x^{1/3} + 1)(5x^4 + 2x)$ • $y'(1) = (1 + 1)(\frac{1}{3}) + (1 + 1)(5 + 2) = 14\frac{2}{3}$

Example 9 Find the rate of change of the area of a rectangle at a moment when the width is 4 meters, the height is 2 meters, the width is increasing 3 meters per hour, and the height is increasing 5 meters per hour.

Answer: $\dfrac{dA}{dt} = w\dfrac{dh}{dt} + h\dfrac{dw}{dt} = [4 \text{ meters}]\left[5\, \dfrac{\text{meters}}{\text{hour}}\right] + [2 \text{ meters}]\left[3\, \dfrac{\text{meters}}{\text{hour}}\right] = 4(5) + 2(3) = 26\, \dfrac{\text{square meters}}{\text{hour}}$

Related-rate problems

In Example 9 we started with an equation, $A = wh$ relating three functions of time, w, h, and A. We differentiated the equation with respect to t to obtain an equation, $A' = wh' + hw'$, which we used to find the rate of change of A from w, h, and their rates of change. This type of problem, which involves rates of change of related functons, is called a RELATED-RATE PROBLEM. Here is another example.

Example 10 At the beginning of 1990 the total population of the U.S was 248.7 million, of whom 51.3% were women, the total population was increasing at the rate of 3.5 million per year, and the percentage of women was decreasing 0.04% per year (data adapted from *the Statistical Abstract of the United States*). At what rate was the population of women increasing at the beginning of 1990?

Answer: Let $p = p(t)$ be the total U.S. population (measured in millions) in year t and let $F = F(t)$ be the fraction that were women (the percent divided by 100). • $p(1990) = 248.7, F(1990) = 0.513$, $p'(1990) = 3.5$, and $F'(1990) = -0.0004$ • [Population of women at time t] $= W(t) = F(t)p(t)$ • $W'(t) = F(t)p'(t) + F'(t)p(t)$ • $W'(1990) = F(1990)p'(1990) + F'(1990)p(1990) = (0.513)(3.5) + (248.7)(-0.0004) \doteq 1.7$ million per year

The Quotient Rule

Derivatives of quotients of functions are found by using the following result.

Theorem 5 (The Quotient Rule) *At any value of x where $y = f(x)$ and $y = g(x)$ have derivatives and $g(x)$ is not zero, $y = f(x)/g(x)$ also has a derivative and*

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\dfrac{df}{dx} - f\dfrac{dg}{dx}}{g^2}. \tag{10}$$

Formula (10) in prime notation is

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}.$$

Remember the Quotient Rule as the following statement: The derivative of a quotient equals the denominator multiplied by the derivative of the numerator, minus the numerator multiplied by the derivative of the denominator, all divided by the square of the denominator.

To prove the theorem, we suppose that $y = f(x)$ and $y = g(x)$ have derivatives at a fixed x and that $g(x) \neq 0$. We let Δx denote a small, nonzero change in the variable, let $\Delta f = f(x + \Delta x) - f(x)$ and $\Delta g = g(x + \Delta x) - g(x)$ be the resulting changes in f and g, and write f for $f(x)$ and g for $g(x)$. Then $f(x + \Delta x) = f + \Delta g$ and $g(x + \Delta x) = g + \Delta g$, and

$$\begin{aligned}
\frac{d}{dx}\left(\frac{f}{g}\right) &= \lim_{\Delta x \to 0} \frac{\dfrac{f + \Delta f}{g + \Delta g} - \dfrac{f}{g}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\frac{g(f + \Delta f) - f(g + \Delta g)}{g(g + \Delta g)}\right] \\
&= \lim_{\Delta x \to 0} \frac{1}{g(g + \Delta g)}\left[\frac{fg + g\Delta f - fg - f\Delta g}{\Delta x}\right] \\
&= \lim_{\Delta x \to 0} \frac{1}{g(g + \Delta g)}\left[g\frac{\Delta f}{\Delta x} - f\frac{\Delta g}{\Delta x}\right].
\end{aligned}$$

Because $g'(x)$ exists, the function g is continuous at x and $\Delta g \to 0$ as $\Delta x \to 0$. Also $\dfrac{\Delta f}{\Delta x} \to \dfrac{df}{dx}$ and $\dfrac{\Delta g}{\Delta x} \to \dfrac{dg}{dx}$ as $\Delta x \to 0$. Therefore the last equation gives

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{1}{g^2}\left[g\frac{df}{dx} - f\frac{dg}{dx}\right].$$

This gives (10) to prove the theorem.

Example 11 What is $S'(0)$ if $R(x) = P(x)/Q(x)$, $P(0) = 3$, $P'(0) = 10$, $Q(0) = 5$, and $Q'(0) = 50$?

Answer: $R'(0) = \dfrac{Q(0)P'(0) - P(0)Q'(0)}{[Q(0)]^2} = \dfrac{5(10) - 3(50)}{5^2} = \dfrac{50 - 150}{25} = -4.$

Example 12 What is $\dfrac{dy}{dx}$ if $y = \dfrac{x^2}{x^4 + a}$ with constant a?

Answer: $\dfrac{dy}{dx} = \dfrac{(x^4 + a)\dfrac{d}{dx}(x^2) - x^2\dfrac{d}{dx}(x^4 + a)}{(x^4 + a)^2} = \dfrac{(x^4 + a)(2x) - x^2(4x^3)}{(x^4 + a)^2} = \dfrac{2ax - 2x^5}{(x^4 + a)^2}$

Example 13 Figures 7 and 8 show graphs of the U.S. national debt $D = D(t)$ and the U. S. population $P = P(t)$ as functions of the time (data adapted from *the Statistical Abstract of the United States*). Find **(a)** the approximate debt per person and **(b)** the approximate rate of increase with respect to time of the debt per person at the beginning of 1985.

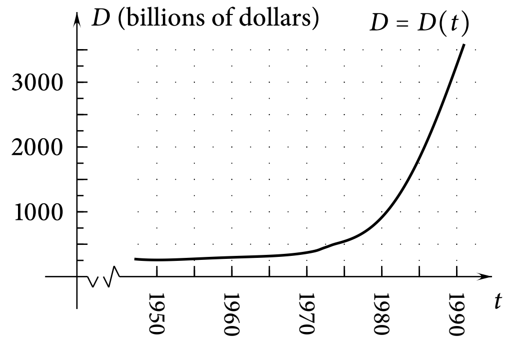


FIGURE 7

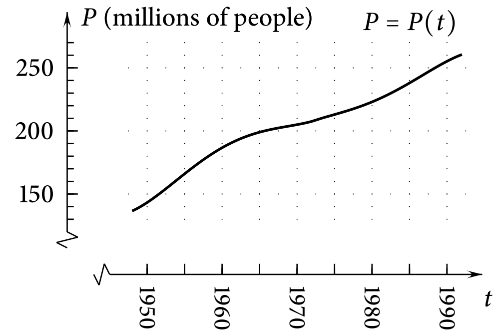


FIGURE 8

Answer: Possible answers: **(a)** $D(1985) \approx 1850$ billion dollars • $P(1985) \approx 240$ million people. •
[Debt per person at the begining of 1985]= $\dfrac{D}{P} \approx \dfrac{1850 \text{ billion dollars}}{240 \text{ million people}} \doteq 7.7$ thousand dollars per person •
(b) The points $(1980, 600)$ and $(1985, 1850)$ are on the approximate tangent line in Figure 9. •
$D'(1985) \approx \dfrac{1850 - 600}{1985 - 1980} \doteq 250$ billion dollars per year • The points $(1970, 180)$ and $(1985, 240)$ are on the approximate tangent line in Figure 10. • $P'(1985) \approx \dfrac{240 - 180}{1985 - 1970} = 4$ million people per year •
[Rate of change of the debt per person] = $\dfrac{d}{dt}\left[\dfrac{D}{P}\right] = \dfrac{PD' - DP'}{P^2} \approx \dfrac{(240)(250) - (1850)(4)}{240^2}$
$\doteq 0.913$ thousand dollars per person per year

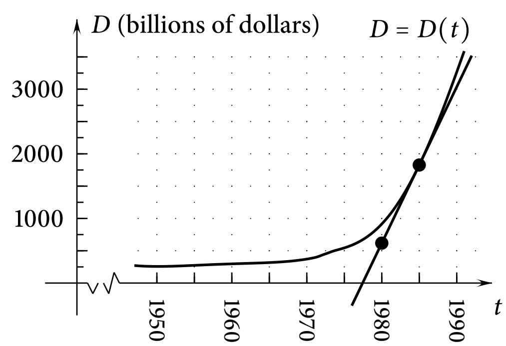


FIGURE 9

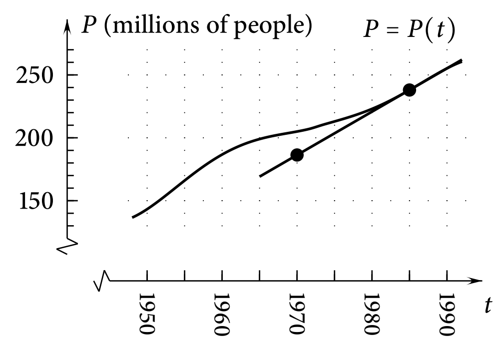


FIGURE 10

Example 14 What is $\dfrac{dy}{dx}$ for $y = \dfrac{\sqrt{x}}{x^5 + 1}$?

Answer: $= \dfrac{(x^5+1)(\frac{1}{2}x^{-1/2}) - x^{1/2}(5x^4)}{(x^5+1)^2}$

Example 15 What is $W'(4)$ if $W(x) = \dfrac{Y(x)}{Z(x)}$, $Y(4) = 2$, $Z(4) = 5$, $Y'(4) = 3$, and $Z'(4) = 6$

Answer: $\frac{3}{25}$

Example 16 Figures 11 and 12 give the graphs of differentiable functions $y = A(x)$ and $y = B(x)$. Give approximate values of AB, A/B, and of their first derivatives at $x = 2$.

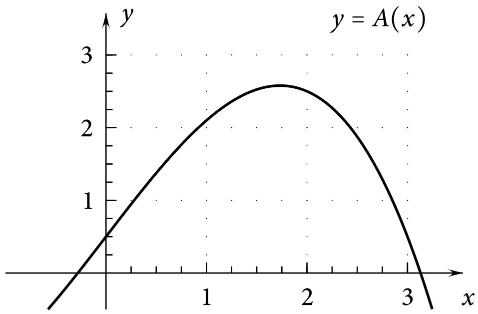


FIGURE 11

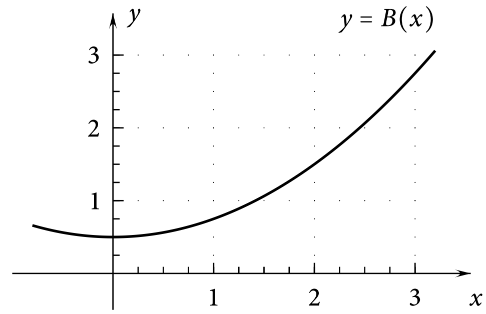


FIGURE 12

Answer: Figure 13 • $A(2) \approx 2.5$ • $A'(2) \approx -0.6$ • Figure 14 • $B(2) \approx 1.5$ • $B'(2) \approx 1$ • $[AB]_{x=2} \approx 3.75$ • $\left[\frac{A}{B}\right]_{x=2} \approx 1.7$ • $\left[\frac{d}{dx}(AB)\right]_{x=2} \approx 1.6$ • $\left[\frac{d}{dx}\left(\frac{A}{B}\right)\right]_{x=2} \approx -1.5$

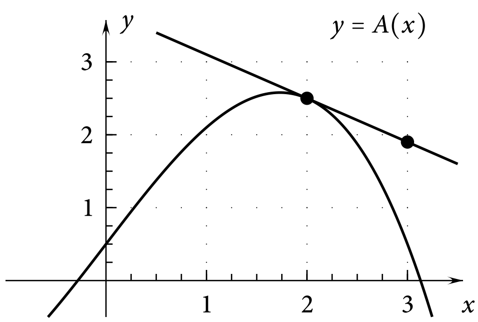


FIGURE 13

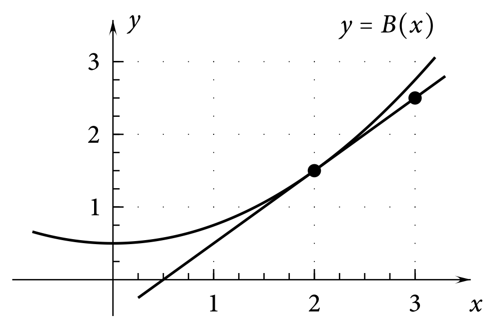


FIGURE 14

Interactive Examples

Work the following Interactive Examples on the class web page, http//www.math.ucsd.edu/~ ashenk/
(The chapter and section numbers on this site do not match the chapters and sections of the textbook.)

Section 2.4: 1–4

Section 2.6: 1–5