Math 20A. Lecture 5 (corrected).

The Chain Rule for powers

We start with the rule for differentiating powers of functions.

Theorem 1 (The Chain Rule for powers)

Suppose that the derivative $\frac{df}{dx}$ of a function f exists at a point x and that n is a constant such that the value $f(x)$ of the function is in an open y-interval where y^{n-1} is defined. Then the derivative of $y = f^n$ at x is

$$\left[\frac{d}{dx}(f^n)\right]_x = n[f(x)]^{n-1}\left[\frac{df}{dx}\right]_x. \tag{1}$$

Remember (1) as the following statement: The derivative of the nth power of a function equals n, multiplied by the $(n-1)$st power of the function, multiplied by the derivative of the function.

To derive (1) we start with the definition,

$$\left[\frac{d}{dx}(f^n)\right]_x = \lim_{\Delta x \to 0} \frac{[f(x+\Delta x)]^n - [f(x)]^n}{\Delta x}.$$

For nonzero Δx, we let Δf denote the change in f when x is changed from x to $x + \Delta x$:

$$\Delta f = f(x+\Delta x) - f(x).$$

We write f for $f(x)$, so that $f(x+\Delta x) = f + \Delta f$. To avoid a complicated step in the general proof of (1), we assume that Δf is not zero for all small nonzero Δx. Then for such Δx,

$$\frac{[f(x+\Delta x)]^n - [f(x)]^n}{\Delta x} = \frac{(f+\Delta f)^n - f^n}{\Delta x} = \left[\frac{(f+\Delta f)^n - f^n}{\Delta f}\right]\left[\frac{\Delta f}{\Delta x}\right]. \tag{2}$$

We obtained the last expression by multiplying and dividing by Δf.

The difference quotient $\frac{\Delta f}{\Delta x}$ on the right of (2) tends to the derivative $\left[\frac{df}{dx}\right]_x$ as $\Delta x \to 0$. Also, Δf tends to 0 as $\Delta x \to 0$ since f is continuous at x, so the difference quotient $\frac{(f+\Delta f)^n - f^n}{\Delta f}$ tends to $\frac{d}{df}(f^n)$, which equals nf^{n-1} by the differentiation rule from the last lecture. Thus equation (2) becomes equation (1) when $\Delta x \to 0$.

Example 1 Find the x-derivative of $y = (x^3+1)^5$.

Answer: $\frac{d}{dx}[(x^3+1)^5] = 5(x^3+1)^4\frac{d}{dx}(x^3+1) = 5(x^3+1)^4(3x^2) = 15x^2(x^3+1)^4$

Example 2 What is $z'(0)$ if $z(x) = [y(x)]^4$, $y(0) = 2$ and $y'(0) = -10$?

Answer: $z' = \frac{d}{dx}(y^4) = 4y^3\,y'$ • $z'(0) = 4[y(0)]^3\,y'(0) = 4(2)^3(-10) = -320.$

Example 3 **(a)** Express the rate of change dV/dt with respect to time of the volume $V = \frac{4}{3}\pi r^3$ of a sphere in terms of the radius r and the rate of change dr/dt of the radius. **(b)** Give a geometric interpretation of the result of part (a). **(c)** At a particular moment, the radius of a sphere is 3 inches and is increasing at rate of 2 inches per minute. How fast is the volume of the sphere increasing at that moment?

Answer: **(a)** $\frac{dV}{dt} = \frac{d}{dt}(\frac{4}{3}\pi r^3) = \frac{4}{3}\pi(3r^2)\frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$ **(b)** The rate of change of the volume of the sphere equals its surface area $4\pi r^2$ multiplied by the rate of change of the radius. **(c)** $r = 3$ • $\frac{dr}{dt} = 5$ • $\frac{dV}{dt} = 4\pi(3^2)(2) = 72\pi$ cubic inches per minute.

Example 4 Figure 1 shows the graph of the area $A = A(t)$ of a square as a function of the time t. Find **(a)** the approximate width w of the square at $t = 6$ and **(b)** the approximate rate of change of the width with respect to time at $t = 6$.

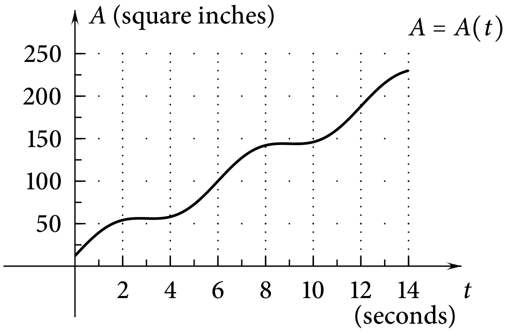


FIGURE 1

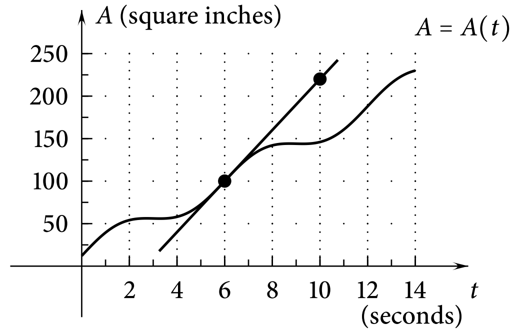


FIGURE 2

Answer: **(a)** $A(6) \approx 100$ cubic inches • $A = w^2$ • $w = \sqrt{A}$ • $w(6) \approx \sqrt{100} = 10$ inches
(b) Figure 2 • Points with approximate coordinates $(6, 100)$ and $(10, 220)$ are on the approximate tangent line.
• At $t = 6$: $\frac{dA}{dt} \approx \frac{220 - 100}{10 - 6} = \frac{120 \text{ square inches}}{4 \text{ seconds}} = 30$ square inches per second • $\frac{dA}{dt} = \frac{d}{dt}(w^2) = 2w\frac{dw}{dt}$ • $30 \approx 2(10)\frac{dw}{dt}$ • $\frac{dw}{dt} \approx \frac{30}{20} = \frac{3}{2}$ inches per second

On the order of operations

Differentiating a complicated expression may require a combination of the Product Rule, the Quotient Rule, the Chain Rule for powers, and other operations. You can determine the order in which to apply these operations by noting the order of the steps used in calculating values of the function. The differentiation is carried out in the reverse order using the Product, Quotient, and Chain Rules to differentiate products, quotients, and powers of functions.

Example 5 Find the x-derivative of the function $y = \left(\frac{x+1}{x-3}\right)^5$.

Answer: Because the last step in finding a value of $y = \left(\frac{x+1}{x-3}\right)^5$ is the taking of the fifth power, we find its derivative by first using the Chain Rule for differentiating the fifth power of a function. • Because the next to last step involves calculating a quotient, the Quotient Rule is used next. • Because the first steps are evaluating $x + 1$ and $x - 3$ these functions are differentiated last. •

$$\frac{d}{dx}\left[\left(\frac{x+1}{x-3}\right)^5\right] = 5\left(\frac{x+1}{x-3}\right)^4 \frac{d}{dx}\left(\frac{x+1}{x-3}\right) = 5\left(\frac{x+1}{x-3}\right)^4 \left[\frac{(x-3)\frac{d}{dx}(x+1) - (x+1)\frac{d}{dx}(x-3)}{(x-3)^2}\right]$$

$$= 5\left(\frac{x+1}{x-3}\right)^4 \left[\frac{(x-3)-(x+1)}{(x-3)^2}\right] = 5\left(\frac{x+1}{x-3}\right)^4 \left[\frac{-4}{(x-3)^2}\right] = \frac{-20(x+1)^4}{(x-3)^6}$$

Example *6* Find $g'(x)$ for $g(x) = x(x^2+1)^{10}$. Do not simplify the answer.

Answer: The Product Rule is used first in differentiating g because the last step in calculating its value involves multiplicatiion. The Chain Rule for differentiating 10th powers of functions is used next because the next to last step in calculating its value involves taking a 10th power. •

$$g'(x) = \frac{d}{dx}[x(x^2+1)^{10}] = x\frac{d}{dx}[(x^2+1)^{10}] + (x^2+1)^{10}\frac{d}{dx}(x)$$
$$= x[10(x^2+1)^9]\frac{d}{dx}(x^2+1) + (x^2+1)^{10}(1) = 10x(x^2+1)^9(2x) + (x^2+1)^{10}$$

Example *7* Express the derivative of $y(x) = [x^2+u(x)]^{3/2}$ in terms of $x, u(x)$, and $u'(x)$.

Answer: $y'(x) = \frac{d}{dx}\{[x^2+u(x)]^{3/2}\} = \frac{3}{2}[x^2+u(x)]^{1/2}\frac{d}{dx}[x^2+u(x)] = \frac{3}{2}[x^2+u(x)]^{1/2}[2x+u'(x)]$

The general Chain Rule

The general Chain Rule deals with derivatives of general composite functions.

Theorem 2 (The Chain Rule) *If $u = u(x)$ has an x-derivative x at x and $y = G(u)$ has a u-derivative at $u(x)$, then $y = G(u(x))$ has an x-derivative at x, which is given by*

$$\left[\frac{d}{dx}[G(u(x))]\right]_x = \left[\frac{dG}{du}\right]_{u(x)}\left[\frac{du}{dx}\right]_x. \tag{3}$$

If we denote the composite function $y = G(u(x))$ by $(G \circ u)(x)$, we can express (3) with prime notation as

$$(G \circ u)'(x) = G'(u(x))\, u'(x) \tag{4}$$

where — since a prime is used to denote the derivative of a function with respect to its own variable — the primes on the left and right denote x-derivatives and the prime in the middle denotes a u-derivative.

To derive (3), we let Δx denote a nonzero change in x from x to $x + \Delta x$ and denote the corresponding changes in $u(x)$ and $G(u(x))$ by Δu and ΔG:

$$\Delta u = u(x+\Delta x) - u(x)$$
$$\Delta G = G(u(x+\Delta x)) - G(u(x)).$$

To avoid a complicated step in the derivation of formula (3), we assume that Δu is not zero for small nonzero Δx. Then for small nonzero Δx,

$$\frac{G(u(x+\Delta x)) - G(u(x))}{\Delta x} = \frac{\Delta G}{\Delta x}.$$

We multiply and divide the last ratio by Δu to obtain

$$\frac{\Delta G}{\Delta x} = \left[\frac{\Delta G}{\Delta u}\right]\left[\frac{\Delta u}{\Delta x}\right]. \tag{5}$$

Because u is continuous at x, Δu tends to zero when Δx tends to zero, and the last equation gives

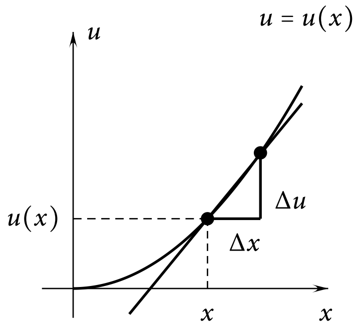
$$\lim_{\Delta x \to 0}\frac{\Delta G}{\Delta x} = \left[\lim_{\Delta u \to 0}\frac{\Delta G}{\Delta u}\right]\left[\lim_{\Delta x \to 0}\frac{\Delta u}{\Delta x}\right].$$

This, by the definition of the derivative, equals

$$\left[\frac{d}{dx}[G(u(x))]\right]_x = \left[\frac{dG}{du}\right]_{u(x)}\left[\frac{du}{dx}\right]_x.$$

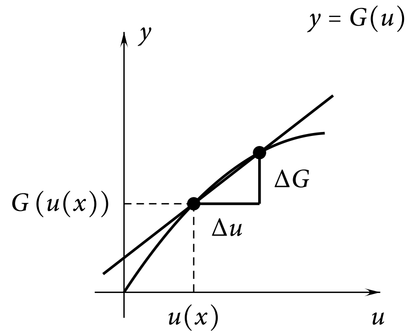
This gives formula (3).

The geometric interpretation of equation (5) is indicated in Figures 3 through 5. The slope $\Delta G/\Delta x$ of the secant line to the graph of the composite function $y = G(u(x))$ in Figure 5 is the product of the slopes $\Delta G/\Delta u$ and $\Delta u/\Delta x$ of the secant lines to the graphs of $u = u(x)$ and $y = G(u)$ in Figures 3 and 4.



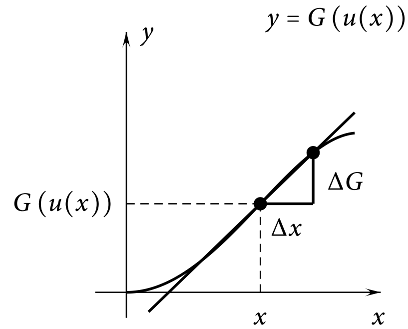
[Slope] = $\dfrac{\Delta u}{\Delta x}$

FIGURE 3



[Slope] = $\dfrac{\Delta G}{\Delta u}$

FIGURE 4



[Slope] = $\dfrac{\Delta G}{\Delta x}$

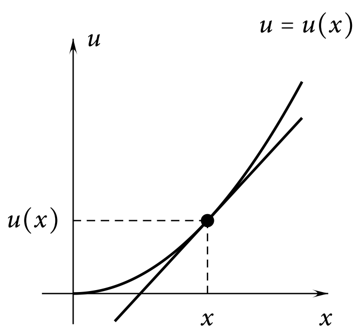
FIGURE 5

When Δx tends to zero, the secant lines in Figures 3 through 5 approach the tangent lines in Figures 6 through 8. These drawings give a geometric interpretation of the Chain Rule itself: If we write $\dfrac{dG}{dx}$ for the slope of the tangent line to the graph of the composite function in Figure 8, then the Chain Rule reads

$$\frac{dG}{dx} = \frac{dG}{du}\frac{du}{dx}. \tag{6}$$

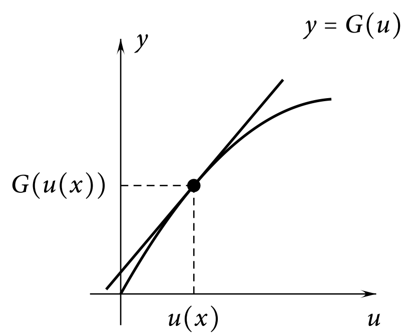
This formula is easy to remember because the terms "du" in the symbolic fractions on the right appear to cancel. It states that the slope of the tangent line to the graph of the composite function in Figure 8 is equal to the product of the slopes of the tangent lines to the graphs of the other functions in Figures 6 and 7.

Equation (6) also has another important interpretation: It states that the rate of change of the variable G with respect to x equals its rate of change with respect to u, multiplied by the rate of change of u with respect to x.



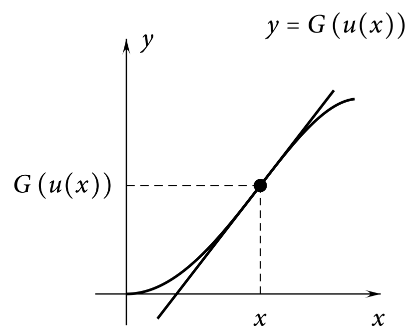
[Slope] = $\dfrac{du}{dx}$

FIGURE 6



[Slope] = $\dfrac{dG}{du}$

FIGURE 7



[Slope] = $\dfrac{dG}{dx}$

FIGURE 8

Example 8 What is the derivative of $y = G(u(x))$ at $x = 5$ if $u(5) = 100$, $\left[\frac{dG}{du}\right]_{u=100} = 7$, and $\left[\frac{du}{dx}\right]_{x=5} = -20$?

Answer: $\left[\frac{d}{dx}[G(u(x))]\right]_{x=5} = \left[\frac{dG}{du}\right]_{u=u(5)} \left[\frac{du}{dx}\right]_{x=5} = \left[\frac{dG}{du}\right]_{u=100} \left[\frac{du}{dx}\right]_{x=5} = (7)(-20) = -140$

Example 9 What is $H'(2)$ if $H(x) = (g \circ u)(x)$, $u(2) = 5$, $u'(2) = -3$, and $g'(5) = 10$?

Answer: $H'(2) = (g \circ u)'(2) = g'(u(2))\,u'(2) = g'(5)u'(2) = (10)(-3) = -30$

Example 10 What is the x-derivative of $S(x) = R(x^4)$ at $x = 2$ if R is a function of u such that $R'(16) = 2$?

Answer: $S'(x) = \frac{d}{dx}[R(x^4)] = R'(x^4)\frac{d}{dx}(x^4) = 4x^3 R'(x^4)$ • $S'(2) = 4(2^3)R'(2^4) = 32\,R'(16) = 32(2) = 64$

The Chain Rule in narrative problems

The Chain Rule is often easier to apply in narrative problems with the Leibniz notation of equation **(6)**,

$$\frac{dG}{dx} = \frac{dG}{du}\frac{du}{dx}.$$

Example 11 At 12:00 PM a balloon is 200 meters above the ground, it is rising 3 meters per second and its volume is increasing at the rate of 0.001 liters per meter of height. At what rate is its volume increasing with respect to time at 12:00 PM?

Answer: Let h be the balloon's height (meters) and V its volume (liters). • At 12:00 PM: $\frac{dh}{dt} = 3$ meters per second, $\frac{dV}{dh} = 0.001$ liters per meter, and $\frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt} = \left[0.001\ \frac{\text{liters}}{\text{meter}}\right]\left[3\ \frac{\text{meters}}{\text{second}}\right] = 0.003$ liters per second

Using the Chain Rule with graphs

The crew and passengers in airplanes are supplied pure oxygen to maintain enough oxygen in their blood when they fly at high altitudes in unpressurized cabins. (A person whose blood oxygen concentration is 50% of its maximum will lose consciousness in about five minutes and can go into a coma within half an hour.) The lower graph in Figure 9 gives the percent $P_A(h)$ of a person's maximum oxygen concentration in his or her blood when he or she is breathing the ambient air in an unpressurized cabin at an altitude of h thousand feet above sea level. The upper curve gives the percent $P_O(h)$ of the maximum oxygen concentration in his blood if he is breathing pure oxygen at altitude h. (Data adapted from *Textbook of Medical Physiology*.)

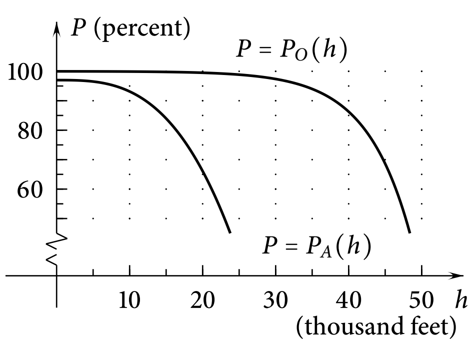


FIGURE 9

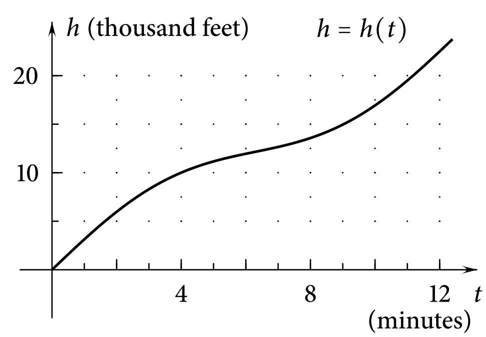


FIGURE 10

Example 12 Suppose that a passenger is breathing the ambient air in the unpressurized cabin of an airplane whose altitude $h = h(t)$ at time t (minutes) is given by the function of Figure 10.
(a) Approximately how high is the plane and what is the passenger's approximate blood oxygen concentration at $t = 4$? **(b)** What is the approximate rate of change with respect to time of the passenger's blood oxygen concentration at $t = 4$?

Answer: (a) Figure 10 shows that $h(4) \approx 10$. • Figure 9 then gives $P_A(h(4)) \approx P_A(10) \approx 94$. • The plane's altitude is approximately 10,000 feet and the passenger's blood-oxygen level is approximately 94% at $t = 4$.
(b) The two points on the tangent line in Figure 11 have approximate coordinates $(10, 94)$ and $(30, 70)$. •
$\left[\frac{dP_A}{dh}\right]_{h=10} \approx \frac{70-94}{30-10} = -1.2$ percent per thousand feet •
The points on the tangent line in Figure 12 have approximate coordinates $(4, 10)$ and $(8, 16)$. •
$\left[\frac{dh}{dt}\right]_{t=4} \approx \frac{16-10}{8-4} = 1.5$ thousand feet per minute • Chain Rule: $\left[\frac{dP_A}{dt}\right]_{t=4} = \left[\frac{dP_A}{dh}\right]_{h=h(4)}\left[\frac{dh}{dt}\right]_{t=4}$
$\approx \left[\frac{dP_A}{dh}\right]_{h=10}\left[\frac{dh}{dt}\right]_{t=4} \approx \left[-1.2\ \frac{\text{percent}}{\text{thousand feet}}\right]\left[1.5\ \frac{\text{thousand feet}}{\text{minute}}\right] = -1.8\ \frac{\text{percent}}{\text{minute}}$

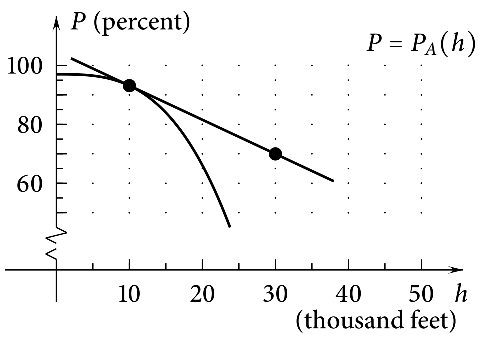


FIGURE 11

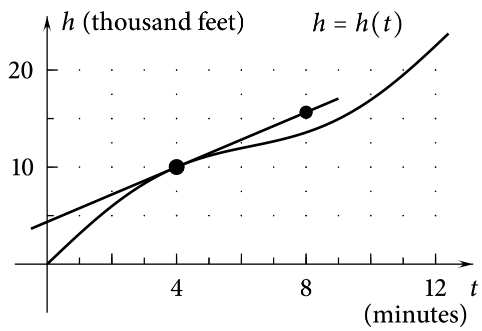


FIGURE 12

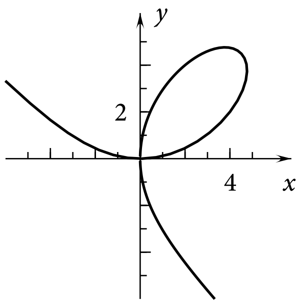
Differentiation of implicitly defined functions

If we cannot solve an equation that defines a function $y = y(x)$ to obtain an explicit formula for the function, we can often find its derivative at any point x_0 where we know its value $y(x_0)$. We take the x-derivative of both sides of the equation that the function satisfies, evaluate the result at x_0 and solve for $y'(x_0)$. This process for finding derivatives of implicitly defined functions is called IMPLICIT DIFFERENTIATION.

Example 13 What is the derivative at $x = 4$ of the function $y = y(x)$ defined implicitly by the equation $x^3 + y^3 = 9xy$ and the condition $y(4) = 2$?

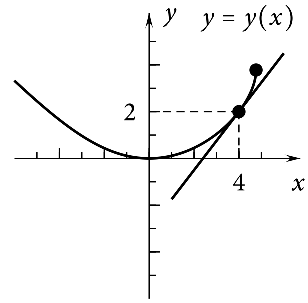
Answer: Use primes to denote x-derivatives. • $(x^3)' + (y^3)' = 9(xy)'$ • Product and Chain Rules:
$3x^2 + 3y^2y' = 9y + 9xy'$ • $x^2 + y^2y' = 3y + 3xy'$ • Set $x = 4$ and $y = 2$. • $4^2 + 2^2y' = 3(2) + 3(4)y'$ where y' denotes $y'(4)$ • $8y' = 10$ • $y'(4) = y' = \frac{10}{8} = \frac{5}{4}$

The graph of the equation $x^3 + y^3 = 9xy$ in Example 13 is the FOLIUM OF DESCARTES in Figure 13. (This curve was studied by the philosopher and mathematician René Descartes in 1638). The graph of the implicitly defined function is in Figure 14. The slope of its tangent line at $x = 4$ is the derivative $y'(4) = \frac{5}{4}$.



$x^3 + y^3 = 9xy$

FIGURE 13



$y = y(x)$

FIGURE 14

Example 14 Find an equation of the tangent line at $(2, 3)$ to the SUPERCIRCLE $x^4 + y^4 = 97$ in Figure 15.

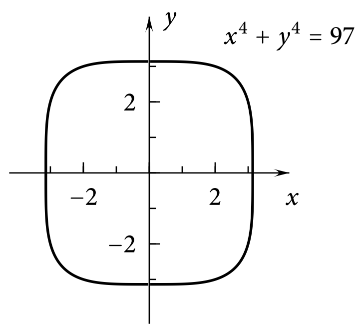


FIGURE 15

Answer: $\frac{d}{dx}(x^4 + y^4) = \frac{d}{dx}(97)$ • $4x^3 + 4y^3y' = 0$ • $x^3 + y^3y' = 0$ • Set $x = 2,\ y = 3 : 27y' + 8 = 0,\ y(2) = 3,\ y'(2) = -\frac{8}{27}$ • Tangent line: $y = 3 - \frac{8}{27}(x - 2)$

Inverse functions

The next theorem shows how to use the derivative of a function to find the derivative of its inverse.

Theorem 1 *Suppose that the domain of $Y = Y(x)$ includes an open interval where $Y'(x)$ is either positive or negative and that $X = X(y)$ is the inverse of $Y = Y(x)$. Then for y in the interval,*

(7)
$$X'(y) = \frac{1}{Y'(X(y))}.$$

Equation (7) in Leibniz notation with $y = Y(x)$ and $x = X(y)$ reads

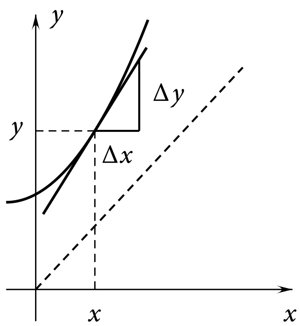
$$\frac{dx}{dy} = \frac{1}{dy/dx}. \tag{8}$$

This relationship between the symbolic fractions $\dfrac{dy}{dx}$ and $\dfrac{dx}{dy}$ is easy to remember because it looks like a rule for ordinary fractions.

To show why (7) holds, suppose that the graph $y = Y(x)$ is the curve in the xy-plane of Figure 16. The graph of its inverse $x = X(y)$ in the yx-plane of Figure 17 is obtained by reflecting the curve and the axes in Figure 16 about the dashed line that bisects the positive x- and y-axes. Figure 16 shows the tangent line to $y = Y(x)$ at x and Figure 17 shows its mirror image, which is the tangent line to $x = X(y)$ at $y = Y(x)$. We suppose that $\Delta x \neq 0$ is the run and Δy is the corresponding rise on the tangent line in Figure 16 so that its slope $y'(x)$ equals $\Delta y/\Delta x$. Then Δy is the run and Δx is the rise in Figure 17 so that

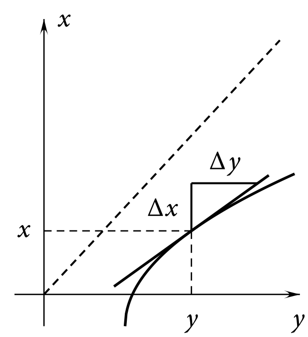
$$X'(y) = \frac{\Delta x}{\Delta y} = \frac{1}{\Delta y/\Delta x} = \frac{1}{Y'(x)}.$$

This is formula (7).



$y = Y(x)$

FIGURE 16



$x = X(y)$

FIGURE 17

Example 15 What is $X'(10)$ if $x = X(y)$ is the inverse of $y = Y(x)$, $Y(2) = 10$, and $Y'(2) = \frac{1}{3}$?

Answer: $X(10) = 2$ because $Y(2) = 10$ • $X'(10) = \dfrac{1}{Y'(X(10))} = \dfrac{1}{Y'(2)} = \dfrac{1}{\frac{1}{3}} = 3$.

Example 16 Suppose that a truck uses $G = G(s)$ gallons of gasoline to go s miles on a trip and that when it has gone 50 miles it has used 3 gallons of gasoline and is consuming gasoline at the rate of $\frac{1}{15}$ gallon per mile. Let $s = s(G)$ denote the inverse of $G = G(s)$. Find the values of s and ds/dG at $G = 3$ What do they denote?

Answer: $s(3) = 50$ (miles) is the distance the truck has traveled when it has used 3 gallons of gas. •
$\dfrac{ds}{dG} = \dfrac{1}{dG/ds} = \dfrac{1}{\frac{1}{15}\text{ gallon per mile}} = 15$ miles per gallon is the rate of which gas is being used when the the truck has used 3 gallons of gas.

Interactive Examples

Work the following Interactive Examples on the class web page, http//www.math.ucsd.edu/~ ashenk/
(The chapter and section numbers on this site do not match those in the textbook for the class.)

Section 3.1: 1, 2, 3

Section 5.3: 1, 2, 4–7