Math 20A. Lecture 6.

Compound interest

Theorem 1 *Suppose that deposit of B_0 dollars is made at at time $t = 0$ (years) in a bank account that pays P% annual interest compounded n times a year. Let $r = \dfrac{P}{100}$ denote the fraction that corresponds to P%. At time $t = \dfrac{1}{n}, \dfrac{2}{n}, \dfrac{3}{n}, \ldots$ the balance is*

(1) $$B = B_0\left(1 + \frac{r}{n}\right)^{nt} \quad \text{dollars.}$$

Every one-nth of a year the fraction r/n of the current balance is paid in interest and the balance is increased by the factor $\left(1 + \dfrac{r}{n}\right)$. After t years this has occurred nt times, raising the balance to the amount in formula (1).

Example 1 According to legend, the Dutch colonist Peter Minuit paid \$24 to buy Manhattan from the Indians in 1626. How much would this investment have become as of year 2010 if it had been kept in a savings account that paid 5% annual interest compounded semi-annually?

Answer: Let $t = 0$ correspond to 1626. • Then $t = 2010 - 1626 = 384$ in year 2000. Use (1) with $B_0 = 24, r = 0.05, n = 2$, and $t = 384$. • The investment would have grown to $24\left(1 + \dfrac{0.05}{2}\right)^{2(384)} = 24(1.025^{768}) \doteq 4.13 \times 10^9$ or approximately 4 billion dollars.

The number e

We number e, which is as important to the calculus of exponential functions and logarithms as the number π is to trigonometry, is defined as the limit,

(2) $$e = \lim_{t\to\infty}\left(1 + \frac{1}{t}\right)^t.$$

It is difficult to prove that this limit exists because the base $(1 + 1/t)$ in the expression $(1 + 1/t)^t$ decreases toward 1 as the exponent t increases toward ∞. The proof is given in advanced calculus courses. The number e has the decimal expansion,

$$e = 2.71828182846\ldots$$

Formula (2) has the following generalization:

Theorem 2 *For any real number x,*

(3) $$e^x = \lim_{t\to\infty}\left(1 + \frac{x}{t}\right)^t.$$

To derive (3) from (2) for a fixed positive x, we set $s = t/x$. Then $t = sx$, $x/t = 1/s$, and $s \to \infty$ as $t \to \infty$, so that by (2)

$$\lim_{t\to\infty}\left(1 + \frac{x}{t}\right)^t = \lim_{s\to\infty}\left(1 + \frac{1}{s}\right)^{sx} = \left[\lim_{s\to\infty}\left(1 + \frac{1}{s}\right)^s\right]^x = e^x.$$

Equation (3) can be derived similarly for $x < 0$ and is valid at $x = 0$, where both sides equal 1.

Interest compounded continuously

The limit as $n \to \infty$ of the balance (1) with interest is compounded n times a year is referred to as the balance when interest is COMPOUNDED CONTINUOUSLY:

$$\begin{bmatrix}\text{Balance at time } t \text{ with interest} \\ \text{compounded continuously}\end{bmatrix} = \lim_{n\to\infty}\left[B_0\left(1+\frac{r}{n}\right)^{nt}\right].$$

We can find this limit by using Theorem 2.

Theorem 3 *Suppose that a deposit of B_0 dollars is made at time $t = 0$ (years) in an account that pays P% annual interest compounded continuously. Set $r = \frac{1}{100}P$. Then the balance at time $t > 0$ is*

$$B(t) = B_0 e^{rt} \text{ dollars.} \tag{4}$$

To derive this formula we write

$$\lim_{n\to\infty}\left[B_0\left(1+\frac{r}{n}\right)^{nt}\right] = B_0\left\{\lim_{n\to\infty}\left[B_0\left(1+\frac{r}{n}\right)^{n}\right]\right\}^t = B_0(e^r)^t = B_0e^{rt}.$$

Example 2 Suppose that deposit of \$100 is made at $t = 0$ in an account that earns 50% annual interest.
(a) What is the balance at $t = 0, 1, 2, 3, 4$ and 5 if the interest is compounded annually?
(b) What is the balance at time $t > 0$ if the interest is compounded continuous;y?

Answer: **(a)** The balance is increased by the factor 1.5 every year. • $B(0) = 100$ • $B(1) = 100(1.5) = 150$ • $B(2) = 150(1.5) = 225$ • $B(3) = 225(1.5) = 337.5$ • $B(4) = 337.5(1.5) = 506.25$ • $B(5) = 506.25(1.5) = 759.375$ **(b)** $r = \frac{1}{100}(5) = 0.5$ • $B(t) = 100e^{0.5t}$

The smooth curve in Figures 1 and 2 is the graph of the balance $B = 100e^{0.5t}$ from Example 2 with interest compounded continuously. The five horizontal line segments in Figure 1 form the graph of the balance with interest compounded annually ($n = 1$), and the five horizontal line segments in Figure 1 form the graph of the balance with interest compounded semi-annually ($n = 2$). You can visualize that the graph with interest compounded n times a year would approach the smooth curve as $n \to \infty$.

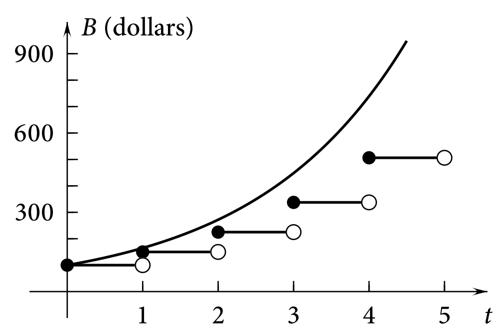


FIGURE 1

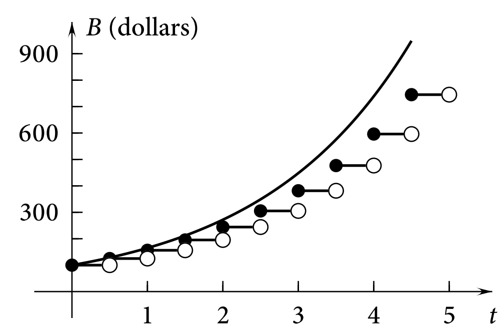


FIGURE 2

Derivatives of logarithms

We will use the definition of the number e to find a formula for derivatives of logarithms.

Theorem 4 *Suppose that b is an arbitrary constant > 1. Then for all $x > 0$,*

$$\frac{d}{dx}(\log_b x) = \frac{\log_b e}{x}. \tag{5}$$

We first use the rule $\log_b(A/B) = \log_b A - \log_b B$ with $A = x + \Delta x$ and $B = x$ to rewrite the difference quotient,

$$\begin{aligned}\frac{\log_b(x+\Delta x) - \log_b x}{\Delta x} &= \frac{1}{\Delta x}[\log_b(x+\Delta x) - \log_b x] \\ &= \frac{1}{\Delta x}\log_b\left(\frac{x+\Delta x}{x}\right) = \frac{1}{\Delta x}\log_b\left(1 + \frac{\Delta x}{x}\right).\end{aligned}$$

Next, we set $\Delta x = x/t$ with a positive constant t, so that

$$\frac{1}{\Delta x} = \frac{t}{x} \quad \text{and} \quad \frac{\Delta x}{x} = \frac{1}{t}$$

and, with the rule $t\log_b(A) = \log_b(A^t)$, we obtain

$$\frac{\log_b(x+\Delta x) - \log_b x}{\Delta x} = \frac{t}{x}\log_b\left(1 + \frac{1}{t}\right) = \frac{1}{x}\log_b\left[\left(1 + \frac{1}{t}\right)^t\right]$$

Because x is fixed and positive, we can have $\Delta x = x/t$ tend to 0 from the right by having t tend to ∞. Then, since $y = \log_b x$ is continuous for $x > 0$,

$$\begin{aligned}\lim_{\Delta x \to 0^+} \frac{\log_b(x+\Delta x) - \log_b x}{\Delta x} &= \lim_{t\to\infty} \frac{1}{x}\log_b\left[\left(1 + \frac{1}{t}\right)^t\right] \\ &= \frac{1}{x}\log_b\left[\lim_{t\to\infty}\left(1 + \frac{1}{t}\right)^t\right] = \frac{\log_b e}{x}.\end{aligned}$$

It can be shown that the same limit is obtained for the one-sided limit as $\Delta x \to 0^-$, so that (5) is valid.

Formula (5) has the simplest form if $b = e$ since $\log_e e = 1$. The logarithm to the base e $y = \log_e x$ is called the NATURAL LOGARITHM and is denoted $y = \ln x$. Theorem 4 gives the following result.

Theorem 5 *For $x > 0$,*

(6) $$\frac{d}{dx}(\ln x) = \frac{1}{x}.$$

Example 3 Find an equation of the tangent line to $y = \ln x$ at $x = 2$.

Answer: Set $y(x) = \ln x$. • Tangent line: $y = y(2) + y'(2)(x-2)$ • $y(2) = \ln(2)$ • $y'(x) = 1/x$ • $y'(2) = \frac{1}{2}$ • Tangent line: $y = \ln(2) + \frac{1}{2}(x-2)$

Instead of using Theorem 1 to find derivatives of the logarithm to the base b, we use Theorem 5 and the rule,

$$\log_b x = \frac{\log_e x}{\log_e b} = \frac{\ln x}{\ln b}.$$

Example 4 What is the rate of change of $y = \log_{10} x$ with respect to x at $x = 3$?

Answer: $y'(x) = \frac{d}{dx}[\log_{10} x] = \frac{d}{dx}\left(\frac{\ln x}{\ln(10)}\right) = \frac{1}{\ln(10)x}$ • $y'(3) = \frac{1}{3\ln(10)}$

Derivatives of logarithms of functions

The Chain Rule from the last lecture and formula (6) yield the following formula for positive differentiable functions $u = u(x)$:

$$\frac{d}{dx}(\ln u) = \frac{d}{du}(\ln u)\,\frac{du}{dx} = \frac{1}{u}\,\frac{du}{dx}. \tag{7}$$

***Example* 5** **(a)** What is the domain of $y = \ln(x^2+1)$? **(b)** What is its x-derivative?

Answer: (a) $y = \ln(x^2+1)$ is defined for all x because $u = x^2+1$ is positive for all x.

(b) $\frac{d}{dx}[\ln(x^2+1)] = \frac{1}{x^2+1}\,\frac{d}{dx}(x^2+1) = \frac{2x}{x^2+1}$

***Example* 6** What is the derivative of $y = x^2\ln(3x+1)$?

Answer: Product and Chain Rules: $\frac{d}{dx}[x^2\ln(3x+1)] = \ln(3x+1)\frac{d}{dx}(x^2) + x^2\frac{d}{dx}[\ln(3x+1)]$

$= 2x\ln(3x+1) + x^2\frac{1}{3x+1}\,\frac{d}{dx}(3x+1) = 2x\ln(3x+1) + \frac{3x^2}{3x+1}$

***Example* 7** The pH of a solution with hydrogen-ion concentration C moles per liter is $y = -\log_{10} C$. A solution contains 5×10^{-9} moles of hydrogen ions per liter at a time when its hydrogen-ion concentration is decreasing 5×10^{-10} moles per liter per hour. At what rate is the pH of the solution increasing or decreasing at that moment?

Answer: Let $C = C(t)$ be the hydrogen-ion concentration at time t. • The pH of the solution at time t is $y = -\log_{10}[C(t)] = \frac{-\ln x}{\ln(10)}$. • $\frac{dy}{dt} = \frac{d}{dt}\left(-\frac{\ln C}{\ln(10)}\right) = -\frac{1}{\ln(10)}\,\frac{d}{dt}(\ln C) = \frac{-1}{C\ln(10)}\,\frac{dC}{dt}$ •

At the moment in question, $C = 5\times10^{-9}$ and $\frac{dC}{dt} = -5\times10^{-10}$ • $\frac{dy}{dt} = \frac{-1}{(5\times10^{-9})\ln(10)}(-5\times10^{-10}) = \frac{1}{10\ln(10)}$

• The pH of the solution is increasing $\frac{1}{10\ln(10)} \doteq 0.043$ pH-units per hour.

Derivative of $y = e^x$

We use the formula for the derivative of the natural logarithm and the rule for finding derivatives of inverse functions from the last lecture to find the derivative of $y = x^x$.

***Theorem* 6** *$y = e^x$ has a derivative for all x, given by*

$$\frac{d}{dx}(e^x) = e^x. \tag{8}$$

Since the inverse of $y = e^x$ is $x = \ln y$

$$\frac{d}{dx}(e^x) = \frac{1}{\frac{d}{dy}(\ln y)} = \frac{1}{1/y} = y = e^x$$

and this is equation (8).

Example 8 Give an equation of the tangent line to $y = e^x$ at $x = 2$ and draw it with the curve.

Answer: Tangent line: $y = y(2) + y'(2)(x-2)$ • $y(2) = e^2$ • $y'(x) = e^x$ • $y'(2) = e^2$ • Tangent line: $y = e^2 + e^2(x-2)$ • Figure 3

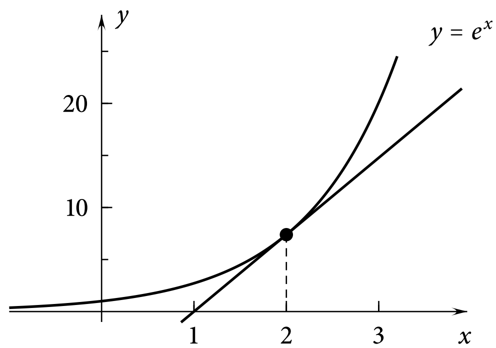


FIGURE 3

The derivative of $y = e^{u(x)}$

Theorem 6 and the Chain Rule imply that if $u = u(x)$ has a derivative at x, then $y = e^{u(x)}$ has a derivative at x, given by

$$\frac{d}{dx}(e^u) = \frac{d}{du}(e^u)\,\frac{du}{dx} = e^u\,\frac{du}{dx}. \tag{9}$$

Remember this result verbally: The x-derivative of e raised to the power u for a function $u = u(x)$ equals e raised to the power u, multiplied by the x-derivative of u.

Example 9 What is the x-derivative of $y = e^{x^3}$?

Answer: $\frac{d}{dx}(e^{x^3}) = e^{x^3}\frac{d}{dx}(x^3) = 3x^2e^{x^3}$

Example 10 Atmospheric pressure at the height h (miles) above the surface of the earth is $p = 1.5e^{-0.2h}$ pounds per square inch. (Data Adapted from the *Encyclopædia Britannica*.) What is the rate of change with respect to time of the atmospheric pressure on a weather balloon when it is one mile in the air and is rising 0.1 miles per hour?

Answer: Let $h = h(t)$ (miles) be the height of the balloon at time t (hours). • $\frac{dp}{dt} = \frac{d}{dt}(1.5e^{-0.2h}) = 1.5e^{-0.2h}\frac{d}{dt}(-0.2h) = -1.5(0.2)e^{-0.2h}\frac{dh}{dt} = -0.3e^{-0.2h}\frac{dh}{dt}$ • At the moment in question, $h = 1$ and $\frac{dh}{dt} = 0.1$ • $\frac{dp}{dt} = -0.3e^{-0.2(1)}(0.1) = -0.03e^{-0.2} \doteq -0.025$ pounds per square inch per hour

Example 11 What is the derivative of $y = \dfrac{e^{-x}}{x}$?

Answer: $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^{-x}}{x}\right) = \dfrac{x\frac{d}{dx}(e^{-x}) - e^{-x}\frac{d}{dx}(x)}{x^2} = \dfrac{xe^{-x}\frac{d}{dx}(-x) - e^{-x}}{x^2} = \dfrac{-xe^{-x} - e^{-x}}{x^2}$

The derivatives of y = sin x and y = cos x

Consider an angle with its vertex at the center of a circle as in Figure 4. Its sides intersect the circle at points P and Q, which are the endpoints of a chord of length $\overline{PQ}$ and an arc of length $\widehat{PQ}$. In order to derive formulas for the derivatives of $\sin x$ and $\cos x$, we need to show that the ratio of these numbers tends to 1 as the points come together:

$$\lim_{P\to Q} \frac{\overline{PQ}}{\widehat{PQ}} = 1. \tag{10}$$

To see why this is true, consider the case where the chord PQ is one side of an polygon with n equal sides that is inscribed in the circle, as shown in Figure 4 for $n = 6$. The chord length $\overline{PQ}$ is one-nth of the perimeter of the polygon and the arclength $\widehat{PQ}$ is one-nth of the circumference of the circle, so that

$$\begin{aligned}\frac{\overline{PQ}}{\widehat{PQ}} &= \frac{[\text{Perimeter of the } n\text{-sided polygon}]/n}{[\text{Circumference of the circle}]/n} \\ &= \frac{\text{Perimeter of the } n\text{-sided polygon}}{\text{Circumference of the circle}}.\end{aligned}$$

Since the perimeter of the polygon tends to the circumference of the circle as n tends to ∞, this quantity tends to 1 and (10) holds.

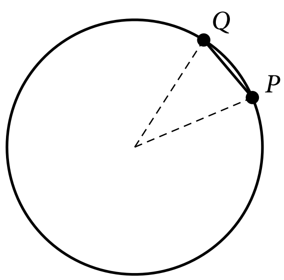


FIGURE 4

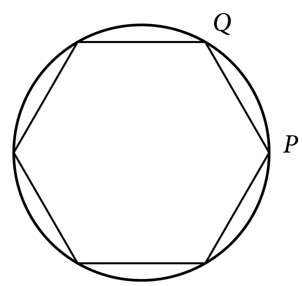


FIGURE 5

Theorem 7 *The functions $y = \sin x$ and $y = \cos x$ have derivatives for all x given by*

$$\begin{aligned}\frac{d}{dx}(\sin x) &= \cos x \\ \frac{d}{dx}(\cos x) &= -\sin x.\end{aligned}$$

To derive these formulas, we consider a fixed, acute angle x and small positive Δx as in Figure 6. The point $P = (\cos x, \sin x)$ is on the unit circle at the angle x and $Q = (\cos(x+\Delta x), \sin(x+\Delta x))$ is on the circle at the angle $x + \Delta x$. Then, with R the vertex at the right angle in the triangle in the drawing,

$$\frac{\sin(x+\Delta x)-\sin x}{\Delta x} = \frac{\overline{PQ}}{\widehat{PQ}}\,\frac{\overline{RQ}}{\overline{PQ}} = \frac{\overline{PQ}}{\widehat{PQ}}\cos\alpha$$
$$\frac{\cos(x+\Delta x)-\cos x}{\Delta x} = -\frac{\overline{PQ}}{\widehat{PQ}}\,\frac{\overline{RP}}{\overline{PQ}} = -\frac{\overline{PQ}}{\widehat{PQ}}\sin\alpha$$

where α is the angle at Q in the right triangle in Figure 6. This angle equals the angle of inclination of the perpendicular bisector OS of the chord PQ because the radius and chord are perpendicular. consequently, α equals $x + \frac{1}{2}\Delta x$ and tends to x as $\Delta x \to 0^+$. Also $\overline{PQ}/\widehat{PQ} \to 1$ by (10), and $\cos\alpha \to \cos x$ and $\sin\alpha \to \sin x$ since these functions are continuous. Therefore,

$$\lim_{\Delta x\to 0^+}\frac{\sin(x+\Delta x)-\sin x}{\Delta x} = \lim_{\Delta x\to 0}\left(\frac{\overline{PQ}}{\widehat{PQ}}\cos\alpha\right) = \cos x$$
$$\lim_{\Delta x\to 0^+}\frac{\cos(x+\Delta x)-\cos x}{\Delta x} = \lim_{\Delta x\to 0}\left(-\frac{\overline{PQ}}{\widehat{PQ}}\sin\alpha\right) = -\sin x.$$

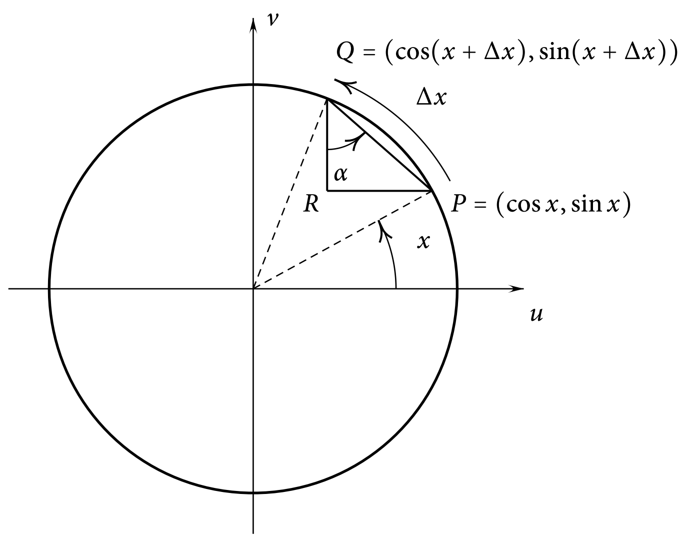


FIGURE 6

Example 12 What is the derivative $\dfrac{d}{dx}(5\sin x - 6\cos x)$?

Answer: $\dfrac{d}{dx}(5\sin x - 6\cos x) = 5\dfrac{d}{dx}(\sin x) - 6\dfrac{d}{dx}(\cos x) = 5\cos x + 6\sin x$

The Chain Rule combined with Theorem 7 yields the following formulas for the x-derivatives of $y = \sin u$ and $y = \cos u$, where u is a differentiable function of x:

$$\frac{d}{dx}(\sin u) = \frac{d}{du}(\sin u)\,\frac{du}{dx} = \cos u\,\frac{du}{dx}$$
$$\frac{d}{dx}(\cos u) = \frac{d}{du}(\cos u)\,\frac{du}{dx} = -\sin u\,\frac{du}{dx}.$$

Example 13 Find the derivative of $y = \sin(x^{1/2})$.

Answer: $\dfrac{dy}{dx} = \dfrac{d}{dx}[\sin(x^{1/2})] = \cos(x^{1/2})\dfrac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{-1/2}\cos(x^{1/2})$

Example 14 What is $y'(10)$ for $y = \cos(u(x))$ if $u(10) = \frac{1}{2}\pi$ and $u'(10) = 3$?

Answer: $y' = (\cos u)' = -\sin u\, u'$ • At $x = 10$: $y'(10) = -\sin(u(10))\,u'(10) = -\sin(\frac{1}{2}\pi)(3) = -3$

Example 15 Suppose that the top of a ten-foot-long ladder, as in Figure 7, is sliding down a vertical wall in such a way that the instantaneous rate of change of the angle x between the horizontal ground and the top of the ladder is -0.3 radians per minute when the angle is $x = 1.1$ radians. How fast is the top of the ladder falling at that moment?

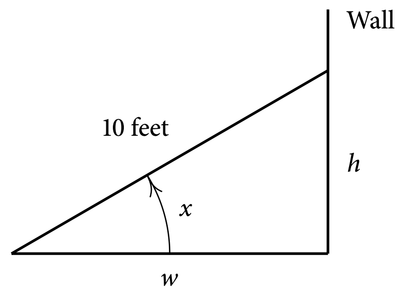


FIGURE 7

Answer: $\sin x = \dfrac{h}{10}$ • $h = 10\sin x$ • $\dfrac{dh}{dt} = \dfrac{d}{dt}(10\sin x) = 10\cos x\,\dfrac{dx}{dt}$ • $x = 1.1$ and $\dfrac{dx}{dt} = -0.3$ • $\dfrac{dh}{dt} = 10\cos(1.1)(-0.3) = -3\cos(1.1)$ • The top of the ladder is falling at the rate of $3\cos(1.1) \doteq 1.36$ feet per minute.

Derivatives of the tangent, cotangent, secant, and cosecant

The derivatives of $y = \tan x$, $y = \cot x$, $y = \sec x$, and $y = \csc x$ can be found by using the Quotient Rule and the Chain Rule for powers with the formulas for the derivatives of $y = \sin x$ and $y = \cos x$.

Theorem 8 *At all values of x where the denominators are not zero,*

$$\frac{d}{dx}(\tan x) = \sec^2 x$$
$$\frac{d}{dx}(\cot x) = -\csc^2 x$$
$$\frac{d}{dx}(\sec x) = \sec x\ \tan x$$
$$\frac{d}{dx}(\csc x) = -\csc x\ \cot x.$$

These rules follow from the differentiation formulas for the sine and cosine and the Pythagorean identity $\cos^2 x + \sin^2 x = 1$.

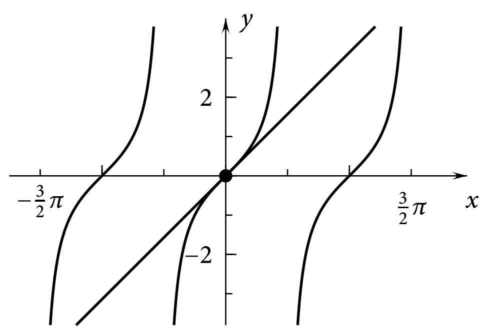
$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left[\frac{\sin x}{\cos x}\right] = \frac{\cos x\dfrac{d}{dx}(\sin x) - \sin x\dfrac{d}{dx}(\cos x)}{\cos^2 x}$$
$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = \frac{d}{dx}\left[\frac{\cos x}{\sin x}\right] = \frac{\sin x \frac{d}{dx}(\cos x) - \cos x \frac{d}{dx}(\sin x)}{\sin^2 x}$$
$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-1}{\sin^2 x} = -\csc^2 x.$$

$$\frac{d}{dx}(\sec x) = \frac{d}{dx}[(\cos x)^{-1}] = -(\cos x)^{-2}\frac{d}{dx}(\cos x)$$
$$= \frac{\sin x}{(\cos x)^2} = \frac{1}{\cos x}\,\frac{\sin x}{\cos x} = \sec x \tan x$$
$$\frac{d}{dx}(\csc x) = \frac{d}{dx}[(\sin x)^{-1}] = -(\sin x)^{-2}\frac{d}{dx}(\sin x)$$
$$= \frac{-\cos x}{(\sin x)^2} = -\frac{1}{\sin x}\,\frac{\cos x}{\sin x} = -\csc x \cot x.$$

Example 16 Give an equation of the tangent line to $y = \tan x$ at $x = 0$.

Answer: Tangent line: $y = f(0) + f'(0)x$ • $f'(x) = \sec^2 x$ • $f(0) = \tan(0) = 0$ and $f'(0) = \sec^2(0) = 1/\cos^2(0) = 1$ • Tangent line: $y = x$ • Figure 7



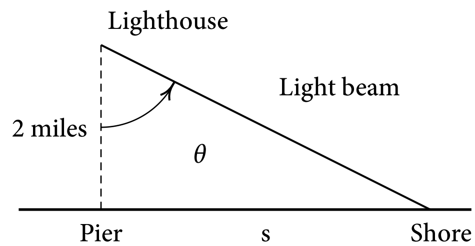
$y = \tan x$
FIGURE 7

Applying the Chain Rule to the formulas in Theorem 8 shows that at any x where $u = u(x)$ has a derivative and no denominator in the formula is zero,

$$\frac{d}{dx}(\tan u) = \frac{d}{du}(\tan u)\,\frac{du}{dx} = \sec^2 u\,\frac{du}{dx}$$
$$\frac{d}{dx}(\cot u) = \frac{d}{du}(\cot u)\,\frac{du}{dx} = -\csc^2 u\,\frac{du}{dx}$$
$$\frac{d}{dx}(\sec u) = \frac{d}{du}(\sec u)\,\frac{du}{dx} = \sec u\,\tan u\,\frac{du}{dx}$$
$$\frac{d}{dx}(\csc u) = \frac{d}{du}(\csc u)\,\frac{du}{dx} = -\csc u\,\cot u\,\frac{du}{dx}.$$

Example 17 A lighthouse is two miles from a pier on a line perpendicular to the straight shore (Figure 8). The beam of light from the lighthouse rotates at the constant rate of three radians per minute. **(a)** Give a formula for the distance s from the pier to the place where the beam of light hits the shore in terms of the angle θ in Figure 8. **(b)** How fast is the beam of light moving along the shore when $\theta = 0.9$ radians?

FIGURE 8



Answer: **(a)** $\tan\theta = \frac{s}{2}$ • $s = 2\tan\theta$

(b) $\frac{ds}{dt} = \frac{d}{dt}(2\tan\theta) = 2\sec^2\theta\frac{d\theta}{dt}$ • Set $\theta = 0.9$ and $\frac{d\theta}{dt} = 3$. • At the moment in question, $\frac{ds}{dt} = 2\sec^2(0.9)(3) = \frac{6}{\cos^2(0.9)} \doteq 15.52$ miles per minute

Derivatives of the inverse sine and tangent

The derivatives of the inverse sine and inverse tangent functions are given in the next theorem.

Theorem 9 *The derivatives of $y = \sin^{-1}x$ and $y = \tan^{-1}x$ are*

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \quad \textit{for } -1 < x < 1$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2} \quad \textit{for all } x.$$

We take x-derivatives of both sides of the identity

$$\sin(\sin^{-1}x) = x \text{ for } -1 < x < 1$$

to obtain

$$\frac{d}{dx}[\sin(\sin^{-1}x)] = \frac{d}{dx}(x)$$

which, with the Chain Rule yields

$$\cos(\sin^{-1}x)\frac{d}{dx}(\sin^{-1}x) = 1.$$

The Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$ then gives

$$\cos(\sin^{-1}x) = \sqrt{1-\sin^2(\sin^{-1}x)} = \sqrt{1-x^2}.$$

Here we use the fact that $-\frac{1}{2}\pi \le \sin^{-1}x \le \frac{1}{2}\pi$, which implies that $\cos(\sin^{-1}x)$ is ≥ 0. Combining the last formulas gives $\sqrt{1-x^2}\frac{d}{dx}(\sin^{-1}x) = 1$ which then gives the formula $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$ for the derivative of the inverse sine function.

Similarly, taking x-derivatives of both sides of the identity $\tan(\tan^{-1} x) = x$ yields

$$\frac{d}{dx}[\tan(\tan^{-1})] = \frac{d}{dx}(x)$$

and then

$$\sec^2(\tan^{-1} x)\frac{d}{dx}(\tan^{-1} x) = 1.$$

The Pythagorean identity $\sec^2\theta = 1 + \tan^2\theta$ enables us to make the substitution,

$$\sec^2(\tan^{-1} x) = 1 + \tan^2(\tan^{-1} x) = 1 + x^2$$

which gives $(1 + x^2)\dfrac{d}{dx}(\tan^{-1} x) = 1$ and then the necessary formula $\dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{1 + x^2}$.

Example 18 What is the derivative of $y = \sin^{-1} x$ at $x = 0$?

Answer: $y'(x) = \dfrac{dy}{dx} = \dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1 - x^2}}$ • $y'(0) = 1$

Example 19 Give an equation of the tangent line to $y = \tan^{-1} x$ at $x = 1$.

Answer: For $y(x) = \tan^{-1} x$, $y(1) = \tan^{-1}(1) = \frac{1}{4}\pi$, $y'(x) = 1/(1 + x^2)$, and $y'(1) = \frac{1}{2}$. The tangent line is $y = \frac{1}{4}\pi + \frac{1}{2}(x - 1)$ (Figure 10)

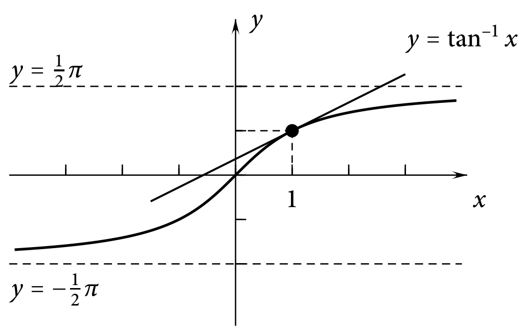


FIGURE 10

The Chain Rule with inverse sine and inverse tangent functions

Theorem 9 and the Chain Rule give, for functions $u = u(x)$,

$$\frac{d}{dx}[\sin^{-1} u] = \frac{1}{\sqrt{1 - u^2}}\,\frac{du}{dx}$$

$$\frac{d}{dx}(\tan^{-1} u) = \frac{1}{1 + u^2}\,\frac{du}{dx}.$$

Example 20 Find the derivatives of **(a)** $y = \sin^{-1}(x^2)$ and **(b)** $y = \left(\sin^{-1} x\right)^3$.

Answer: (a) $\dfrac{d}{dx}[\sin^{-1}(x^2)] = \dfrac{1}{\sqrt{1 - (x^2)^2}}\,\dfrac{d}{dx}(x^2) = \dfrac{2x}{\sqrt{1 - x^4}}$

(b) $\dfrac{d}{dx}\left[\left(\sin^{-1} x\right)^3\right] = 3\left(\sin^{-1} x\right)^2 \dfrac{d}{dx}(\sin^{-1} x) = \dfrac{3\left(\sin^{-1} x\right)^2}{\sqrt{1 - x^2}}$

Example 21 **(a)** Express ψ in Figure 11 in terms of y. **(b)** What is $\frac{d\psi}{dt}$ at a time when $y = 2$ meters and $\frac{dy}{dt} = 3$ meters per minute?

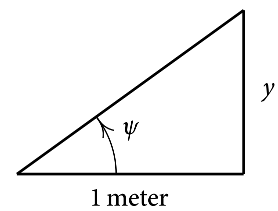


FIGURE 11

Answer: **(a)** $\tan\psi = \frac{y}{1} = y$ • $\psi = \tan^{-1}(y)$ radians **(b)** $\frac{d\psi}{dt} = \frac{1}{1+y^2}\frac{dy}{dt}$ • At the time in question: $\frac{d\psi}{dt} = \frac{1}{1+2^2}(3) = \frac{3}{5}$ radians per minute

Interactive Examples

Work the following Interactive Examples on the class web page, http//www.math.ucsd.edu/∼ ashenk/ (The chapter and section numbers on this site do not match those in the textbook for the class.)

Section 3.2: 1–3

Section 3.3: 3–5

Section 3.5: 1–5

Section 3.6: 1, 2, 4