Math 20A. Lecture 7.

This lecture deals with related rate problems which differ from those in earlier lectures in that the necessary equations relating the functions are harder to find. We will use similar triangles and the Pythagorean Theorem.

Using similar triangles

Example 1 A light on the horizontal ground in Figure 1 is casting the shadow of a 10-foot-high fence on the side of a building. The fence is 8 feet from the building. Let s denote the distance between the light and the fence and let h denote the length of the shadow, as in Figure 1.

(a) Find a formula for the length h in terms of the distance s.

(b) Find an equation relating these functions and their time derivatives.

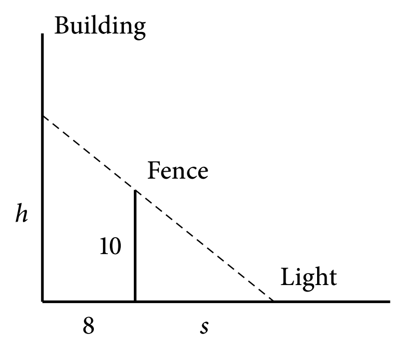

FIGURE 1

Answer: **(a)** The right triangle of height h and base of length $s + 8$ in Figure 1 is similar to the right triangle of height 10 and base of length s. • $\dfrac{h}{8+s} = \dfrac{10}{s}$ • $sh = 10(8 + s)$ • $h = \dfrac{80 + 10s}{s}$ • $h = 80s^{-1} + 10$

(b) Consider s and h to be functions of the time t. • Chain Rule for powers of functions:
$\dfrac{dh}{dt} = \dfrac{d}{dt}(80s^{-1} + 10) = -80s^{-2}\dfrac{ds}{dt} = \dfrac{-80}{s^2}\dfrac{ds}{dt}$

Example 2 When the light in Figure 1 is 10 feet from the fence, it is moving away from the fence at the rate of 5 feet per minute. At what rate is the length of the shadow increasing or decreasing at that moment?

Answer: At the moment in question, $s = 10$ and $\dfrac{ds}{dt} = 5$ • $\dfrac{dh}{dt} = \dfrac{-80}{s^2}\dfrac{ds}{dt} = \dfrac{-80}{10^2}(5) = -4$ • The length of the shadow is decreasing 4 feet per minute.

Example 3 Water is flowing into a tank that has the shape of a right circular cone with its vertex pointed down (Figure 2). The radius of the top of the tank is 10 feet and it is 20 feet high. At what rate is water being added when it is 5 feet deep if, at that moment, the water level is rising at the rate of 4 feet per hour? (The volume of a right circular cone with base of radius r and height h is $V = \frac{1}{3}\pi r^2 h$.)

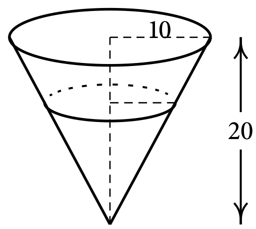

FIGURE 2

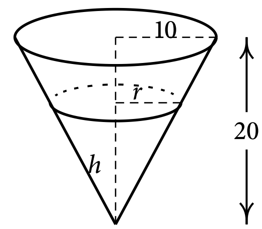

FIGURE 3

Answer: Let $h = h(t)$ be the depth of the water and $r = r(t)$ the radius of the surface of the water (Figure 3). •
Similar triangles: $\dfrac{h}{r} = \dfrac{20}{10}$ • $r = \frac{1}{2}h$ • $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 h = \frac{1}{12}\pi h^3$ •
Chain Rule for powers: $\dfrac{dV}{dt} = \dfrac{d}{dt}\left(\frac{1}{12}\pi h^3\right) = \frac{1}{4}\pi h^2 \dfrac{dh}{dt}$ • $h = 5$ and $\dfrac{dh}{dt} = 4$ • $\dfrac{dV}{dt} = \frac{1}{4}\pi(5)^2(4) = 25\pi$ •
Water is being added at the rate of 25π cubic feet per hour.

Using the Pythagorean theorem

Example 4 A car is being driven east on an east-west highway, away from an intersection of two highways. At the same time a truck is being driven south toward the intersection on a north-south highway. Let $x = x(t)$ denote the car's distance from the intersection, let $y = y(t)$ denote the truck's distance from the intersection, and let $D = D(t)$ denote the distance between the car and truck as in Figure 4. Find **(a)** an equation relating x, y, and D and **(b)** an equation relating these variables and their time derivatives.

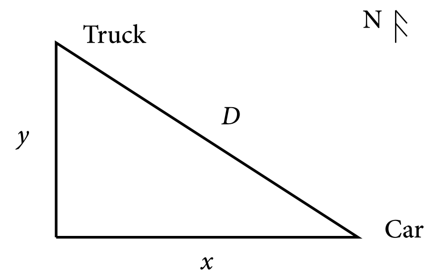

FIGURE 4

Answer: **(a)** One approach: Pythagoran Theorem: $x^2 + y^2 = D^2$

(b) $\frac{d}{dt}(x^2 + y^2) = \frac{d}{dt}(D^2)$ • $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2D\frac{dD}{dt}$ • $x\frac{dx}{dt} + y\frac{dy}{dt} = D\frac{dD}{dt}$

Example 5 When the car in Example 4 is four miles east of the intersection, it is going 50 miles per hour away from it, while the truck is three miles north of the intersection and is traveling 70 miles per hour toward it. At what rate is the distance between the car and the truck increasing or decreasing at that moment?

Answer: From Example 4: $\left\{\begin{array}{c} x^2 + y^2 = D^2 \\ x\frac{dx}{dt} + y\frac{dy}{dt} = D\frac{dD}{dt} \end{array}\right\}$ • Set $x = 4$ and $y = 3$ in the first equation. •
$D^2 = 4^2 + 3^2 = 25$ • $D = 5$ • Set $x = 4, y = 3, D = 5, \frac{dx}{dt} = 50$ and $\frac{dy}{dt} = -70$ in the second equation. •
$5D' = (4)(50) + (3)(-70) = -10$ • $D' = -2$. The distance between the car and the truck is decreasing 2 miles per hour at the moment in question.

Interactive Examples

Work the following Interactive Examples on the class web page, http//www.math.ucsd.edu/~ ashenk/
(The chapter and section numbers on this site do not match those in the textbook for the class.)

Section 5.2: 1–3