Math 20A. Lecture 8.

The First-Derivative Test at a point

A function f is INCREASING AT THE POINT x_0 if there are numbers a and b with $a < x_0 < b$ such that $f(x) < f(x_0)$ for $a < x < x_0$ and $f(x_0) < f(x)$ for $x_0 < x < b$. Similarly, f is DECREASING AT THE POINT x_0 if there are numbers a and b with $a < x_0 < b$ such that $f(x) > f(x_0)$ for $a < x < x_0$ and $f(x_0) > f(x)$ for $x_0 < x < b$.

Theorem 1 (The First-Derivative Test at a point)

(a) *If $f'(x_0)$ exists and is positive, then f is increasing at x_0.*

(b) *If $f'(x_0)$ exists and is negative, then f is decreasing at x_0.*

This result follows easily from the definition of the derivative. Consider part (a). The derivative $f'(x_0)$ is the limit

$$f(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

If this limit is positive, then there are numbers a and b with $a < x_0 < b$ such that $\frac{f(x) - f(x_0)}{x - x_0}$ is positive for $a < x < x_0$ and for $x_0 < x < b$. This implies that $f(x) - f(x_0)$ is negative and $f(x) < f(x_0)$ for $a < x < x_0$ since $x - x_0$ is negative there. This also implies that $f(x) - f(x_0)$ is positive and $f(x) > f(x_0)$ for $x_0 < x < b$ where $x - x_0$ is positive. Hence f is increasing at x_0. Part (b) of the the theorem can be established with a similar argument.

Example 1 Show that $y = \sin x$ is increasing at all numbers x with $0 < x < \frac{1}{2}\pi$.

Answer: $y = \sin x$ is increasing at any x with $0 < x < \frac{1}{2}\pi$ since its derivative $\cos x$ is positive for all such x.

Local maxima and minima: a necessary condition

A function f has a LOCAL MAXIMUM at a point x_0 if $f(x_0)$ is the greatest value of f for x in an open interval containing x_0. The function has a LOCAL MAXIMUM at x_0 if $f(x_0)$ is the least value of f for x in such an interval. The function f of Figure 1 has a local maximum at one negative x and a local minimum at one positive x.

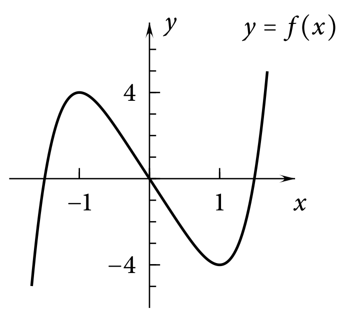


FIGURE 1

A number x_0 is called a CRITICAL POINT or a CRITICAL VALUE of a function f if it is in an open interval in the domain of f and the derivative $f'(x_0)$ at x_0 is zero or does not exist.

Theorem 2 (A First-Derivative Test for local maxima and minima) *If $y = f(x)$ has a local maximum or local minimum at $x = x_0$, then x_0 is a critical point of f.*

This theorem holds because if $f'(x_0)$ exists and is not zero, then f is either increasing or decreasing at x_0 and cannot have a local maximum or local minimum there.

Example 2 The curve in Figure 1 above is the graph of $f(x) = x^5 - 5x$. Find its local maximum and local minimum and the points where they occur.

Answer: The local maximum and minimum occur at critical points of the function $f(x) = x^5 - 5x$ • $f'(x) = 5x^4 - 5 = 5(x^4 - 1)$ is defined for all x and zero where $x^4 - 1 = 0$. • Solve $x^4 - 1 = 0$ to find the critical points. • The critical points are $x = 1$ and $x = -1$. • The graph shows that there is a local maximum at $x = -1$, where $f(x) = (-1)^5 - 5(-1) = 4$ and a local minimum at $x = 1$, where $f(x) = 1^2 - 5(1) = -4$.

Example 3 The function $y = x - 2\ln x$, whose graph is shown in Figure 2, has a local minimum. Find its value and where it occurs.

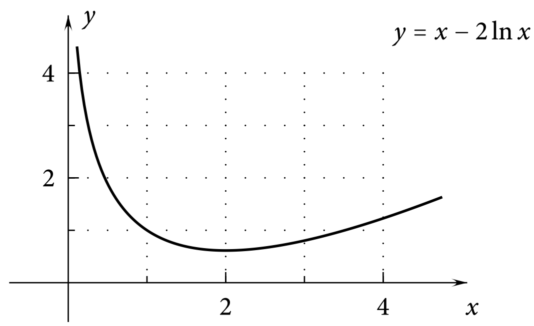


FIGURE 2

Answer: $y = x - 2\ln x$ • For $x > 0$, $y'(x) = \dfrac{d}{dx}(x - 2\ln x) = 1 - \dfrac{2}{x} = \dfrac{x-2}{x}$ • This is zero only at $x = 2$. • The only critical point is 2, so the function's local minimum must occur there. • The value of the local minimum is the value $y(2) = 2 - 2\ln(2) \doteq 0.6137$ of the function.

Because the local minimum $y(2) = 2 - 2\ln(2)$ is the least value for all x of the function in Figure 2, it is also the GLOBAL MINIMUM of the function.

Example 4 The function $y = 3 - |x|$ of Figure 3 has a local maximum at $x = 0$. Explain why $x = 0$ is a critical point of the function.

Answer: The function does not have a derivative at $x = 0$ because the difference quotient $\dfrac{y(x) - y(0)}{x - 0} = \dfrac{(3 - |x|) - 3}{x} = \dfrac{-|x|}{x}$ equals -1 for $x > 0$ and equals 1 for $x < 0$, so it does not have a two-sided limit as $x \to 0$. (You might expect that the function does not have a derivative at $x = 0$ because the curve is too pointed for it to have a tangent line there.)

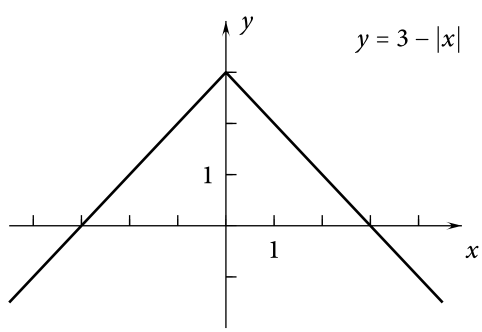


FIGURE 3

The Extreme Value and Mean Value Theorems

In order to discuss the next topic, we need two geometrically plausible results.

Theorem 3 (The Extreme Value Theorem) *If a function f is continuous on a closed finite interval $[a, b]$, then f has a maximum value and a minimum value on the interval.*

This theorem states that under the given conditions there is a highest and a lowest point on the graph $y = f(x)$ for $a \le x \le b$.

Theorem 4 (The Mean Value Theorem) *Suppose that $y = f(x)$ is continuous on the finite closed interval $[a, b]$ and that $f'(x)$ exists for all x with $a < x < b$. Then there is at least one number c with $a < c < b$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \tag{1}$$

The derivative $f'(c)$ on the left of equation (1) is the slope of the tangent line to the graph of f at $x = c$. The difference quotient $\dfrac{f(b) - f(a)}{b - a}$ on the right is the slope of the secant line between the points at $x = a$ and $x = b$. The theorem states under the given conditions there is a point c with $a < c < b$ such that the tangent line and secant line are parallel as in Figire 4.

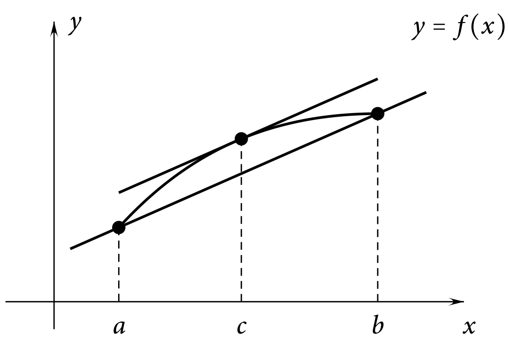


FIGURE 4

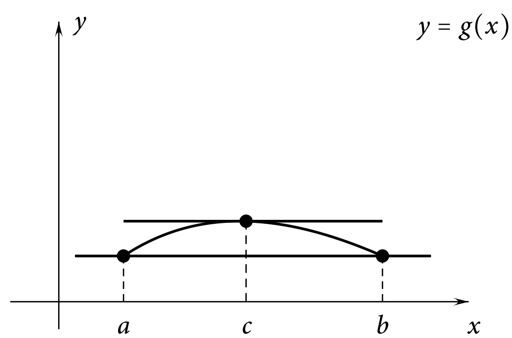


FIGURE 5

To verify the Mean Value Theorem, we let m denote the slope $\dfrac{f(b) - f(a)}{b - a}$ of the secant line and define a new function g by $g(x) = f(x) - mx$ (Figure 5). This function is also continuous on $[a, b]$. It has the same value at $x = a$ and $x = b$ because

$$\begin{aligned} g(b) - g(a) &= [f(b) - mb] - [f(a) - ma] \\ &= [f(b) - f(a)] - m(b - a) \\ &= [f(b) - f(a)] - [f(b) - f(a)] = 0. \end{aligned}$$

Consequently, the secant line through the points at $x = a$ and $x = b$ on the graph of g is horizontal, as is shown in Figure 5.

The funcion g has a maximum and a minimum value on $[a,b]$ by the Extreme Value Theorem. If the maximum is greater than $g(a) = g(b)$, then it occurs at a point c with $a < c < b$ and is a local maximum so that $g'(c) = 0$. If the minimum is less than $g(a) = g(b)$, then it occurs at a point c with $a < c < b$ and is a local minimum so that $g'(c) = 0$. If the maximum and minimum are both equal to $g(a) = g(b)$, then $g(x)$ is constant and $g'(c) = 0$ for all c with $a < c < b$.

In any case, there is a number c with $a < c < b$ such that $g'(c) = 0$. Since

$$g'(x) = \frac{d}{dx}[f(x) - mx] = f'(x) - m$$

this implies that $f'(c) = m$ and gives (1) to establish the thoerem.

Increasing and decreasing functions on an interval

A function f is INCREASING ON AN INTERVAL if $f(x)$ increases as x increases across the interval. This means that $f(a) < f(b)$ for all a and b in the interval with $a < b$. A function f is DECREASING ON AN INTERVAL if $f(x)$ decreases as x increases across the interval. This means that $f(a) > f(b)$ for all a and b in the interval with $a < b$.

Theorem 5 (The First-Derivative Test on an interval)

(a) *If $y = f(x)$ is continuous on an interval and $f'(x)$ exists and is positive at all points in the interior of the interval, then $y = f(x)$ is increasing on the interval.*

(b) *If $y = f(x)$ is continuous on an interval and $f'(x)$ exists and is negative at all points in the interior of the interval, then $y = f(x)$ is decreasing on the interval.*

To see why this result is valid, suppose that f is continuous on an interval I and that $f'(x)$ exists for all x in the interior of I. Consider any points a and b in I with $a < b$. The function f is continuous on $[a,b]$ and $f'(x)$ exists for $a < x < b$, so that by the Mean Value Theorem, there is a number c with $a < c < b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \tag{3}$$

If $f'(x)$ is positive for x in the interior of I, then $f'(c)$ and $b - a$ are positive and equation (3) shows that $f(b) - f(a)$ is positive. Hence, $f(b) > f(a)$ and, since a and b are arbitrary points in I with $a < b$, f is increasing on I, as asserted in part (a) of the theorem.

If $f'(x)$ is negative for x in the interior of I, then the number (3) is negative. This implies that $f(b) < f(a)$ for any a and b in I with $a < b$, so that f is decreasing on I, as stated in part (b).

***Example* 5** Find the intervals on which $f(x) = x^3 - 3x^2$ is increasing and decreasing.

Answer: $f(x) = x^3 - 3x^2$ is defined and continuous for all x. •
$f'(x) = \frac{d}{dx}(x^3 - 3x^2) = 3x^2 - 6x = 3x(x-2)$ is zero at $x = 0$ and at $x = 2$. • Figure 6 •
For $x < 0$, the derivative $3x(x-2)$ is positive because x and $x - 2$ are negative. •
f is increasing on $(-\infty, 0]$. •
For $0 < x < 2$, the derivative $3x(x-2)$ is negative because x is positive and $x - 2$ is negative. •
f is decreasing on $[0, 2]$. •
For $x > 2$, the derivative $3x(x-2)$ is positive because x and $x - 2$ are positive. •
f is increasing on $[2, \infty)$.

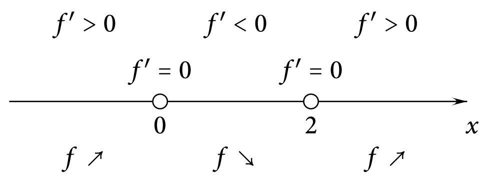


FIGURE 6

***Example* 6** **(a)** Find the limits of the function $f(x) = x^3 - 3x^2$ of Example 5 as $x \to \infty$ and as $x \to -\infty$.
(b) Use this information and the results of Example 5 to sketch the graph of f.

Answer: (a) The limits as $x \to \pm\infty$ of $f(x) = x^3 - 3x^2$ are the same as those of its highest degree term x^3 because $|x^3|$ is much larger than x^2 for large x. • Algebraic justification: For $x \neq 0$, $x^3 - 3x^2 = x^3\left(1 - \dfrac{3}{x}\right)$ • $\lim\limits_{x\to\infty}(x^3 - 3x^2) = \lim\limits_{x\to\infty} x^3 = \infty$ • $\lim\limits_{x\to\infty}(x^3 - 3x^2) = \lim\limits_{x\to-\infty} x^3 = -\infty$

(b) Plot the points at $x = 0$ and $x = 2$, where the derivative is zero. • $f(0) = 0^3 - 3(0^2) = 0$ • $f(2) = 2^3 - 3(2^4) = -4$ • Draw the curve through these points so it tends to $-\infty$ as $x \to -\infty$, is increasing on $(-\infty, 0]$, is decreasing on $[0, 2]$, is increasing on $[2, \infty)$, and tends to ∞ as $x \to \infty$. • Figure 7

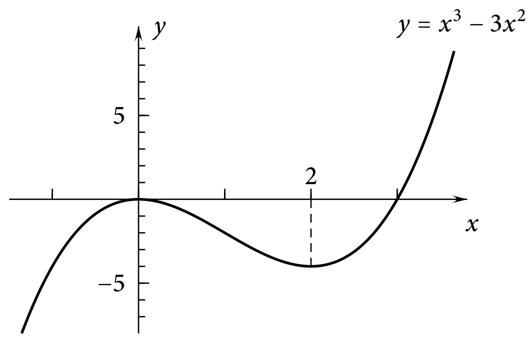
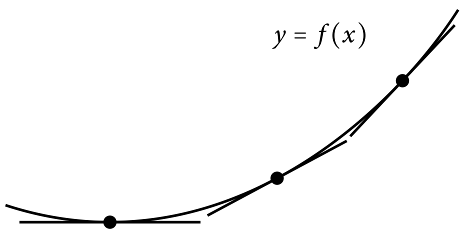


FIGURE 7

Notice that the graph of $y = x^3 - 3x^2$ looks like the curve $y = -3x^3$ for small $|x|$. This is because $|x^3|$ is much smaller than $3x^2$ for very small $|x|$.

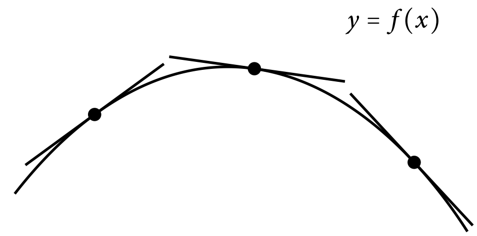
The Second-Derivative Test for concavity

The portion of the graph $y = f(x)$ for x in an open interval is CONCAVE UP if the slope $f'(x)$ of the tangent line at x to the graph increases as x increases across the interval (Figure 8). If the slope $f'(x)$ decreases as x increases across the open interval, then that portion of the graph of the function is CONCAVE DOWN (Figure 9).



Slope increasing
Graph concave up

FIGURE 8



Slope decreasing
Graph concave down

FIGURE 9

The intervals in which the graph of a function is concave up and the intervals where it is concave down can be found by studying the function's second derivative.

Theorem 1 (The Second-Derivative Test for concavity)

(a) *If $f''(x)$ exists and is positive on an open interval, then the graph of $y = f(x)$ is concave up on the interval.*

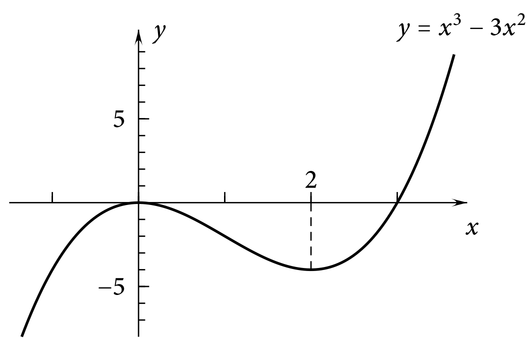
(b) *If $f''(x)$ exists and is negative in an open interval, then the graph of $y = f(x)$ is concave down on the interval.*

This theorem holds because $f'(x)$ is increasing on any open interval were $f''(x)$ is positive and is decreasing on any open interval were $f''(x)$ is negative.

***Example** 7* Find the open intervals where the graph of the function $y = x^3 - 3x^2$ of Examples 5 and 6 is concave up and concave down.

Answer: $f(x) = x^3 - 3x^2$ • $f'(x) = 3x^2 - 6x$ • $f''(x) = 6x - 6 = 6(x-1)$ • $f''(x)$ is negative for $x < 1$ and positive for $x > 1$. • The graph is concave down on $(-\infty, 1)$ and concave up on $(1, \infty)$ • Figure 10

FIGURE 10



A point where a curve switches from concave up to concave down or vise versa is called an INFLECTION POINT. The point at $x = 1$ in Figure 10 is an inflection point of that curve.

***Example* 8** Figure 11 shows the graph of $f(x) = xe^x$. Explain its shape by determining **(a)** where the function is positive and negative, **(b)** where it is increasing and decreasing, and **(c)** where its graph is concave up and concave down. **(d)** What are the inflection points of the graph?

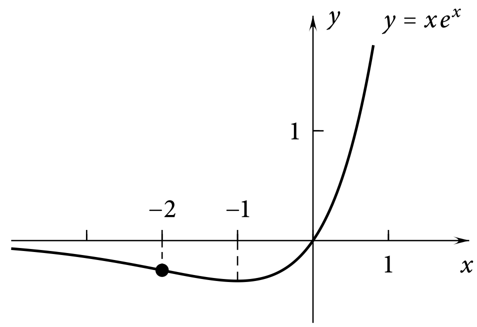


FIGURE 11

Answer: **(a)** $f(x) = xe^x$ is defined and continuous for all x. • It is negative for $x > 0$, is zero at $x = 0$, and is positive for $x > 0$. • The graph is below the x-axis for $x < 0$, passes through the origin, and is above the x-axis for $x > 0$.

(b) Product Rule: $f'(x) = \frac{d}{dx}(xe^x) = x\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x) = (x+1)e^x$ • $f'(x)$ is negative for $x < -1$ and positive for $x > -1$. • f is decreasing on $(-\infty, -1]$ and increasing on $[-1, \infty)$.

(c) $f''(x) = \frac{d}{dx}[(x+1)e^x] = (x+1)\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x+1) = (x+2)e^x$ • $f''(x)$ is negative for $x < -2$ and positive for $x > -2$. • The graph is concave down for $x < -2$ and concave up for $x > -2$.

(d) The point $(-2, -2e^{-2})$ at $x = -2$ on the graph is its one inflection point.

***Example* 9** Draw the graph of $g(x) = 4x^3 - x^4$ by studying the formula for the function, by determing the largest intervals on which the function is increasing and decreasing, and by finding the largest open intervals on which its graph is concave up and concave down. Show any local or global maxima and minima and any inflection points.

Answer: The polynomial $g(x) = 4x^3 - x^4$ is continuous for all x and has the same limits as $x \to \pm\infty$ as its highest order term $y = -x^4$. • $\lim_{x\to\pm\infty} g(x) = \lim_{x\to\pm\infty} (4x^3 - x^4) = \lim_{x\to\pm\infty} (-x^4) = -\infty$ •

$g'(x) = \dfrac{d}{dx}(4x^3 - x^4) = 12x^2 - 4x^3 = 4x^2(3 - x)$ is zero at $x = 0$ and $x = 3$. • Figure 12 • $g'(x)$ is positive for $x < 0$ because x^2 and $(3 - x)$ are positive there, • $g'(x)$ is positive for $0 < x < 3$ since x^2 and $3 - x$ are positive there, • $g'(x)$ is negative for $x > 3$ since x^2 is positive and $3 - x$ is negative there. • g is increasing on $(\infty, 0]$ and $[0, 3]$ and therefore on $(-\infty, 3]$. • g is decreasing on $[3, \infty)$. • $g(x)$ has a global maximum at $x = 3$.

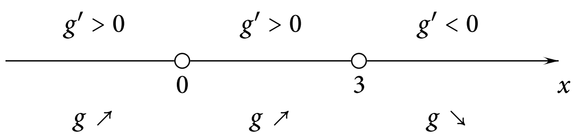


FIGURE 12

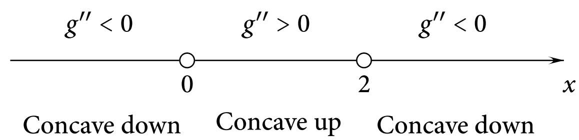


FIGURE 13

$g''(x) = \dfrac{d}{dx}(12x^2 - 4x^3) = 24x - 12x^2 = 12x(2 - x)$ is zero at $x = 0$ and $x = 2$ and has constant sign for $x < 0$, for $0 < x < 2$, and for $x > 2$. • Figure 13 • $g''(-1) = 12(-1)(2 + 1) = -36$ is negative, so $g''(x)$ is negative for $x < 0$. • $g''(1) = 12(1)(2 - 1) = 12$ is positive, so $g''(x)$ is positive for $0 < x < 2$. • $g''(3) = 12(3)(2 - 1) = -36$ is negative, so $g''(x)$ is negative for $x > 2$. • The graph is concave down for $x < 0$ and for $x > 2$, and is concave up for $0 < x < 2$. • $g(0) = 4(0^3) - 0^4 = 0$ • $g(2) = 4(2^3) - 2^4 = 16$ • $g(3) = 4(3^3) - 3^4 = 3^3 = 27$. • Figure 14 • The graph has inflection points at $(0, 0)$ and $(2, 16)$.

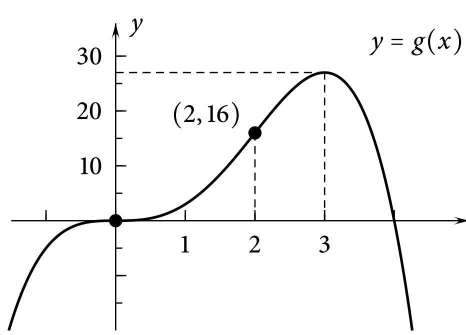


FIGURE 14

Interactive Examples

Work the following Interactive Examples on the class web page, http//www.math.ucsd.edu/~ ashenk/ (The chapter and section numbers on this site do not match those in the textbook for the class.)

Section 4.1: 1–3

Section 4.2: 1–2

Section 4.3: 1–2

Section 4.4: 1