Math 20A. Lecture 9.

Maxima and minima on finite, closed intervals

In the last lecture we found local maxima and minima of functions by determining where their derivatives were positive and where they were negative. In cases where we need to find maxima and minima of a continuous function on a finite, closed interval, we can also use a shorter procedure, described in the following theorem, that only requires finding the critical points and comparing values of the function.

Theorem 1 *If $y = f(x)$ is continuous on a finite closed interval $[a, b]$, then it has a maximum and a minimum value on the interval, and the maximum and minimum occur either at a, at b, or at critical points in the interior (a, b) of $[a, b]$.*

This theorem follows easily from earlier results. If f is continuous on $[a, b]$, then it has a maximum and a minimum value on that interval by the Extreme Value Theorem. The maximum or minimum might occur at an endpoint a or b of the interval. Otherwise, it occurs at a point x_0 in the interior of the interval and is a local maximum or minimum, so that, by Theorem 1 of the last section, x_0 is a critical point of the function.

Example 1 Find the maximum and minimum of $f(x) = 2x^{1/2} - x$ for $0 \le x \le 9$.

Answer: $f(x) = 2x^{1/2} - x$ has a maximum and a minimum for $0 \le x \le 9$ because it is continuous on the finite, closed interval $[0, 9]$. • $f'(x) = \dfrac{d}{dx}(2x^{1/2} - x) = 2(\frac{1}{2}x^{-1/2}) - 1 = x^{-1/2} - 1 = \dfrac{1-\sqrt{x}}{\sqrt{x}}$ exists for $x > 0$ and is zero at $x = 1$. • The one critical point is $x = 1$. • The maximum and minimum on the interval are the greatest and least of $f(0) = 2(0^{1/2}) - 0 = 0$, $f(1) = 2(1^{1/2}) - 1 = 1$, and $f(9) = 2(9^{1/2}) - 9 = -3$. • The maximum is 1 at $x = 1$. • The minimum is -3 at $x = 9$.

The result of Example 1 can be corroborated by the fact that $f'(x) = \dfrac{1-\sqrt{x}}{\sqrt{x}}$ is positive for $0 < x < 1$ and negative for $x > 1$ so that $f(x)$ is increasing on $[0, 1]$ and decreasing on $[1, 9]$.

(7/17/11)

The Second-Derivative Test at a local maximum or minimum

We usually show that a function $y = f(x)$ has a local maximum or minimum at a critical point x_0 by determining whether the derivative is positive or negative in open intervals to the left and to the right of x_0. In some cases, however, it is more convenient to use the following result.

Theorem 2 (The Second-Derivative Test for a local maximum or minimum)

(a) *If $f'(x_0) = 0$ and $f''(x_0) > 0$, then $y = f(x)$ has a local minimum at x_0.*
(b) *If $f'(x_0) = 0$ and $f''(x_0) < 0$, then $y=f(x)$ has a local maximum at x_0.*

Part (a) of this theorem follows easily from earlier results. If $f'(x_0)$ is zero and $f''(x_0)$ is positive, then $f'(x)$ is increasing at x_0, so that $f'(x)$ is negative on an interval (a, x_0) to the left of x_0 and positive on an interval (x_0, b) to the right of x_0. This implies that $f(x)$ is decreasing on $(a, x_0]$ and increasing on $[x_0, b)$ so that it has a local minimum at x_0. Part (b) follows similarly.

Example 2 Use the Second-Derivative Test to show that $h(x) = 3 - 2x + \frac{1}{2}x^2 + \frac{1}{4}x^4$ has a local minimum at $x = 1$.

Answer: $h'(x) = \dfrac{d}{dx}(3 - 2x + \frac{1}{2}x^2 + \frac{1}{4}x^4) = -2 + x + x^3$ • $h'(1) = -2 + 1 + 1^3 = 0$ •
$h''(x) = \dfrac{d}{dx}(-2 + x + x^3) = 1 + 3x^2$ • $h''(1) = 1 + 3(1)^2 = 4$ is positive, so $y = h(x)$ has a local minimum at $x = 1$.

Example 3 A rectangular garden is to be formed using a wall as one side and 40 feet of fence for the three other sides (Figure 1). Find the dimensions that give the garden the maximum area.

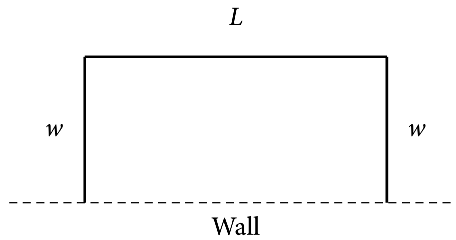


FIGURE 1

Answer: Let L (feet) denote the length ($0 \le L \le 40$) and w (feet) the width ($0 \le w \le 20$) of the garden as in Figure 1. • [Area] $= wL$ • [Length of fence used] $= L + 2w$ • Use all the fence: $L + 2w = 40$ • Solve for L in terms of w. • $L = 40 - 2w$ • $A = wL = w(40 - 2w)$ • $A(w) = 40w - 2w^2$ • $A'(w) = \frac{d}{dw}(40w - 2w^2) = 40 - 4w = 4(10 - w)$ • The derivative is zero at $w = 10$, is positive for $0 < w < 10$, and is negative for $10 < w < 40$. • The area is a maximum for $w = 10$ feet, for which $L = 40 - 2(10) = 20$ feet. • The maximum area is $20(10) - 200$ square feet. • The graph of the area as a function of w is in Figure 2.

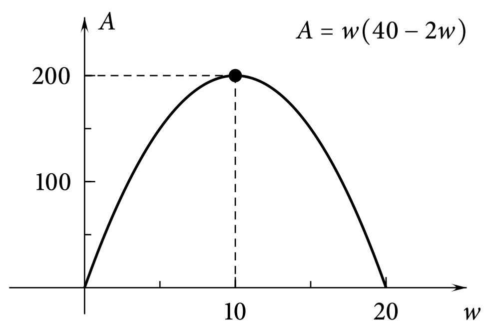


FIGURE 2

Example 4 What dimensions should the garden in Figure 1 have to use the least amount of fence if the area is to be 200 square feet?

Answer: [Area] $= wL = 200$ • $L = \frac{200}{w}$ • [Fence length] $= F = L + 2w = 200w^{-1} + 2w$ • $\frac{dF}{dw} = -200w^{-2} + 2 = \frac{2w^2 - 200}{w^2} = \frac{2(w^2 - 100)}{w^2}$ • $\frac{dF}{dw}$ is zero at $w = 10$, negative for $0 < w < 10$, and positive for $w > 0$. • The length of fence is a minimum if $w = 10$ (feet) and $L = \frac{200}{10} = 20$ (feet).

Example 5 A rectangular box with a square base is to be constructed so that it has a volume of 2 cubic meters (Figure 3). The material for the base and top costs two dollars per square meter and the material for the four sides costs one dollar per square meter. How wide and how tall should the box be to minimize the cost of the material?

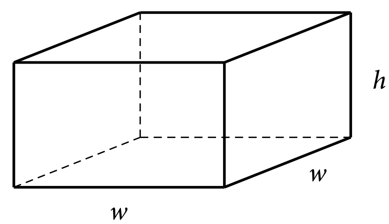


FIGURE 3

Answer: [Total cost] $= C = \left[2w^2 \text{ meters}^2\right]\left[2\ \frac{\text{dollars}}{\text{meter}^2}\right] + \left[4wh \text{ meters}^2\right]\left[1\ \frac{\text{dollar}}{\text{meter}^2}\right] = 4w^2 + 4wh$ dollars • [Volume] $= w^2h = 2$ • $h = \frac{2}{w^2}$ • $C = 4w^2 + \frac{4w(2)}{w^2} = 4w^2 + 8w^{-1}$ • $\frac{dC}{dw} = 8w - 8w^{-2} = \frac{8(w^3-1)}{w^2}$ • $\frac{dC}{dw}$ is zero at $w = 1$, negative for $0 < w < 1$ and positive for $x > 1$. • The cost is a minimum if $w = 1$ (meter) and $h = \frac{2}{w^2} = 2$ (meters).

Example 6 (Not required) In ancient Greek and Roman legends, Queen Dido took her followers ca. 800 BC from Tyre to North Africa, where the local inhabitants said she could have all the land she could enclose in an ox hide. She had the skin cut into fine strips to form a long cord that enclosed a region that grew to be the prosperous city of Carthage. (Later, when a nearby king demanded that Dido marry him, she killed herself rather than give up her city. Later Virgil (70–19 BC) changed the story in the *Aeneid* by having Dido commit suicide when Aeneas left her.) This legend led modern mathematicians to refer to the following question as *Dido's problem:* Of all closed curves of the same length, which encloses the largest area? What is the answer?

Answer: The circle

Example 7 Find the point on the curve $y = \sqrt{x^2+1}$ in Figure 4 that is closest to the point $(2,0)$. (Minimize the square of the distance from $(2,0)$ to the point at x on the curve.)

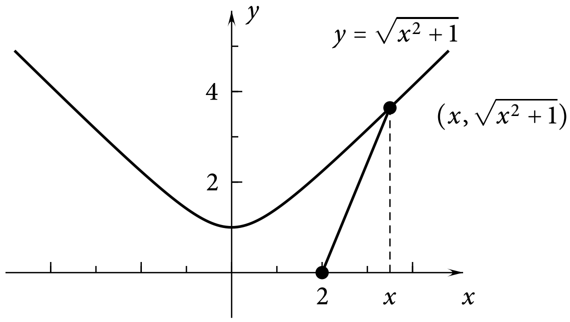


FIGURE 4

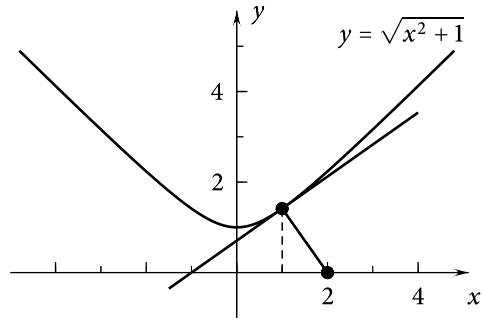
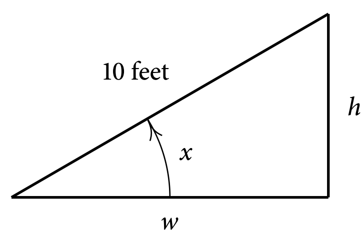


FIGURE 5

Answer: The point on the curve with x-coordinate x is $(x, \sqrt{x^2+1})$ (Figure 4) • Pythagorean theorem: $F(x) = [\text{Distance}]^2 = (x-2)^2 + (\sqrt{x^2+1} - 0)^2 = 2x^2 - 4x + 5$ • $F'(x) = 4x - 4 = 4(x-1)$ • $F'(x)$ is zero at $x = 1$, negative for $x < 1$, and positive for $x > 1$. • The distance is a minimum at $x = 1$. • The closest point is $(1, \sqrt{1^2+1}) = (1, \sqrt{2})$. • Figure 5

Example 8 **(a)** Figure 6 shows a right triangle with hypotenuse of length 10 feet. Give a formula for the area of the triangle as a function of the angle x in the drawing. Where is this function defined and continuous? **(b)** Of all such right triangles, which has the largest area?

FIGURE 6



Answer: **(a)** Let w be the width and h the height of the triangle. • $w = 10\cos x$, $h = 10\sin x$ • $[\text{Area}] = A(x) = \frac{1}{2}wh = 50\cos x\,\sin x$ • $A(x)$ is defined and continuous on $[0, \frac{1}{2}\pi]$.

(b) Product Rule: $\frac{dA}{dx} = \frac{d}{dx}(50\cos x\sin x) = 50[\cos x\frac{d}{dx}(\sin x) + \sin x\frac{d}{dx}(\cos x)] = 50(\cos^2 x - \sin^2 x)$ •

$\frac{dA}{dx}$ is zero where $\sin^2 x = \cos^2 x$, which is at $x = \frac{1}{4}\pi$, • The maximum of $A(x)$ is the maximum of $A(0) = 0$, $A(\frac{1}{2}\pi) = 0$, and $A(\frac{1}{4}\pi)$, which is positive. • The triangle with maximum area is the 45°-45°-right triangle.

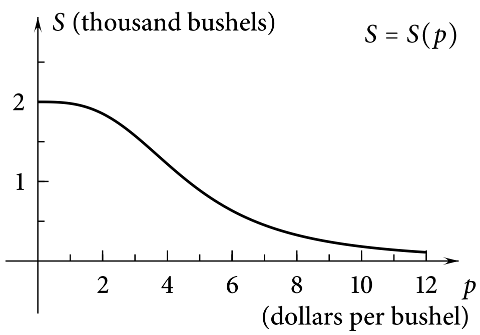
Example 9 If a farmer charges p dollars per bushel for soybeans, he would sell $S(p) = \dfrac{200}{p^3+100}$ thousand bushels of beans. ($S = S(p)$ is called a DEMAND FUNCTION.) The amount of money he would receive from the sales (his REVENUE) is the product $R(p) = pS(p)$ of the demand and the price. Find the price (to the nearest cent) that would maximize the revenue from the soybeans.

Answer: $R(p) = p\,S(p) = \left[p\,\dfrac{\text{dollars}}{\text{bushel}}\right]\left[\dfrac{200}{p^3+100}\text{ thousand bushels}\right] = \dfrac{200p}{p^3+100}$ thousand dollars •

$R'(p) = \dfrac{d}{dp}\left[\dfrac{200p}{p^3+100}\right] = \dfrac{(p^3+100)\dfrac{d}{dp}(200p) - 200p\dfrac{d}{dp}(p^3+100)}{(p^3+100)^2} = \dfrac{200(p^3+100)) - 200p(3p^2)}{(p^3+100)^2}$

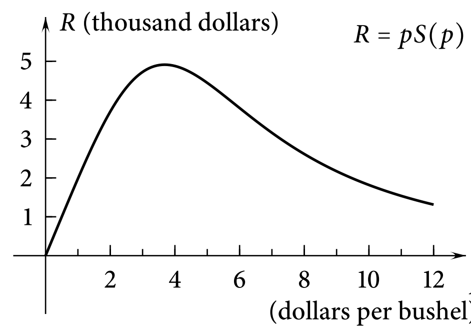
$= \dfrac{400(50-p^3)}{(p^3+100)^2}$ • $R'(p)$ is zero at $p = 50^{1/3}$, positive for $0 < p < 50^{1/3}$, and negative for $p > 50^{1/3}$. •

Its maximum is at $p = 50^{1/3} \doteq 3.68403$. • The optimum price is either \$3.68 or \$3.69 per bushel. A calculator or computer gives $R(3.68) \doteq 4.91204$ and $R(3.69) \doteq 4.91203$ thousand dollars , so the optimum price is \$3.68 per bushel. • The graphs of $S = S(p)$ and $R = R(p)$ are shown in Figures 7 and 8.



The demand function

FIGURE 7



The revenue function

FIGURE 8

Example 10 A train has an average of 1200 passengers per day when its fare is \$2.00 per person. Suppose that for every ten cents that the fare is decreased, down to \$1.50 per person, the train will average 100 more passengers each day. What fare should be charged to maximize the average daily revenue?

Answer: Let x be the number of ten-cent discounts. • R=[Revenue] $= (2 - 0.1x)(1200 + 100x)$ $= -10x^2 + 80x + 2400$ dollars for $x \geq 0$ • $\dfrac{dR}{dx} = 80 - 20x = 20(4 - x)$ is positive for $x < 4$ and decreasing for $x > 4$. • R is a maximum at $x = 4$. • Charge \$1.60.

Interactive Examples

Work the following Interactive Examples on the class web page, http//www.math.ucsd.edu/~ ashenk/
(The chapter and section numbers on this site do not match those in the textbook for the class.)

Section 4.5: 1–2

Section 4.6: 1, 2, 4, 5