Quiz 2 Solutions

1) We must determine the value of $\frac{dh}{dt}$ at the moment when $\frac{dx}{dt}$ is 1/10 radian per minute and $x = \frac{1}{2}$ radian. The variables x and h are related by the equation

$$\sin x = \frac{h}{10}. \tag{1}$$

Differentiating both sides of (1) with respect to t and using the Chain Rule gives

$$\cos x \cdot \frac{dx}{dt} = \frac{1}{10} \cdot \frac{dh}{dt}.$$

Substituting 1/10 for $\frac{dx}{dt}$ and 1/2 for x gives

$$\cos(1/2) \cdot \frac{1}{10} = \frac{1}{10}\frac{dh}{dt}$$

which implies $\frac{dh}{dt} = \cos(1/2)$. The rate of change in the length of h at the moment when $x = 1/2$ radian and the angle x is increasing at 1/10 radians per minute is $\cos(1/2)$ feet per second.

2) The perimeter of a right triangle, one of whose angles is x, is $10 + 10\cos x + 10\sin x$. Let $P(x) = 10 + 10\cos x + 10\sin x$. We need to optimize P over the interval $0 \leq x \leq \frac{\pi}{2}$. We start by taking the derivative of P.

$$P'(x) = -10\sin x + 10\cos x.$$

Next we solve $P'(x) = 0$ where $0 \leq x \leq \frac{\pi}{2}$.

$$\begin{aligned} -10\sin x + 10\cos x &= 0 \\ 10\cos x &= 10\sin x \\ \cos x &= \sin x \\ x &= \frac{\pi}{4} \end{aligned}$$

Evaluating P at the endpoints of the interval and the critical point $x = \pi/4$,

$$P(0) = 20 \qquad P(\tfrac{\pi}{4}) = 10 + 10\tfrac{\sqrt{2}}{2} + 10\tfrac{\sqrt{2}}{2} = 10 + 10\sqrt{2} \qquad P(\tfrac{\pi}{2}) = 20$$

The maximum value of P is at $x = \frac{\pi}{4}$. The right triangle with hypotenuse of length 10 that maximizes the perimeter is an isosceles triangle.

3) (a) Using the formula $\frac{d}{dx}e^x = e^x$, the Power Rule, and linearity of the derivative,

$$\frac{d}{dx}(x^2 + e^x) = 2x + e^x.$$

(b) Using the formula $\frac{d}{dx}\ln x = \frac{1}{x}$ and the Product Rule,

$$f'(x) = x \cdot \frac{d}{dx}\ln x + \ln x \cdot \frac{d}{dx}x = x(1/x) + \ln x \cdot 1 = 1 + \ln x.$$

Therefore $f'(e) = 1 + \ln e = 1 + 1 = 2$.

4) The equation of the tangent line at $x = 1$ is $y = y(1) + y'(1)(x - 1)$. The two values that need to be computed are $y(1)$ and $y'(1)$.

$$y(1) = \tan^{-1}(1) = \frac{\pi}{4}.$$

To compute $y'(1)$, we use the formula $\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$ to obtain

$$y' = \frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$$

so that $y'(1) = \frac{1}{1+1^2} = \frac{1}{2}$. The equation of the tangent line to $y = \tan^{-1} x$ at $x = 1$ is

$$y = \frac{\pi}{4} + \frac{1}{2}(x - 1).$$

5) Let x be the distance from the origin to P, let y be the distance from the origin to Q, and let z be the distance from P to Q. We must determine $\frac{dy}{dt}$ at the moment when Q is at $(0, 12)$, P is at $(5, 0)$, and $\frac{dz}{dt}$ is 10. By the Pythagorean Theorem, $x^2 + y^2 = z^2$. Differentiate both sides of this equation with respect to time t to obtain

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}. \tag{2}$$

When $x = 5$ and $y = 12$,

$$z = \sqrt{5^2 + 12^2} = \sqrt{25 + 144} = \sqrt{169} = 13.$$

Substituting $z = 13$, $x = 5$, $y = 12$, $\frac{dz}{dt} = 10$, and $\frac{dx}{dt} = 2$ into (2) gives

$$2(5)(2) + 2(12)\frac{dy}{dt} = 2(13)(10).$$

Solving this equation for $\frac{dy}{dt}$ gives $\frac{dy}{dt} = 10$. The point Q is moving in the positive y direction at a rate of 10 meters per second.

6) (a) $\lim\limits_{x\to-\infty} x + \dfrac{4}{x} = -\infty$ since $\lim\limits_{x\to-\infty} \dfrac{4}{x} = 0$ so that the behaviour of $f(x)$ for x large is determined by the behaviour of x. Similarly $\lim\limits_{x\to\infty} x + \dfrac{4}{x} = \infty$. On the other hand, when x is small the behaviour of $f(x)$ is determined by $\frac{4}{x}$ since this term grows arbitrarily large as x shrinks to 0 while the term x shrinks to 0. Thus $\lim\limits_{x\to 0^-} x + \dfrac{4}{x} = -\infty$ and $\lim\limits_{x\to 0^+} x + \dfrac{4}{x} = \infty$.

(b) In order to determine on which intervals f is increasing or decreasing we need to determine the sign of f'.

$$f'(x) = 1 - \frac{4}{x^2}.$$

Next we solve $f'(x) = 0$.

$$\begin{aligned} 1 - \frac{4}{x^2} &= 0 \\ 1 &= \frac{4}{x^2} \\ x^2 &= 4 \\ x &= \pm 2 \end{aligned}$$

The number 0 is not in the domain of f and so we partition the real line into 4 intervals: $(-\infty, -2)$, $(-2, 0)$, $(0, 2)$, and $(2, \infty)$. On each of these intervals the sign of $f'(x)$ is the same for each choice of x within a fixed interval. In order to determine the sign of f' we choose the sample point -3 from the interval $(-\infty, -2)$, -1 from the interval $(-2, 0)$, 1 from the interval $(0, 2)$, and 3 from the interval $(2, \infty)$.

$$f'(\pm 3) = 1 - \tfrac{4}{(\pm 3)^2} = 1 - \tfrac{4}{9} = \tfrac{5}{9} > 0 \qquad f'(\pm 1) = 1 - \tfrac{4}{(\pm 1)^2} = -3 < 0$$

Therefore f is increasing on the intervals $(-\infty, -2)$ and $(2, \infty)$ and f is decreasing on the intervals $(-2, 0)$ and $(0, 2)$.

(c) In order to determine the intervals on which f is concave up and concave down we need to determine the sign of f''. From part (b), $f'(x) = 1 - 4x^{-2}$ so that

$$f''(x) = 8x^{-3} = \frac{8}{x^3}.$$

The equation $f''(x) = 0$ has no solutions and so we divide the real line into two intervals $(-\infty, 0)$ and $(0, \infty)$ (recall the value 0 is not in the domain of f). On the interval $(-\infty, 0)$ the sign of f'' stays the same and this is also true for the interval $(0, \infty)$. Choose sample point -1 from the interval $(-\infty, 0)$ and 1 from the interval $(0, \infty)$.

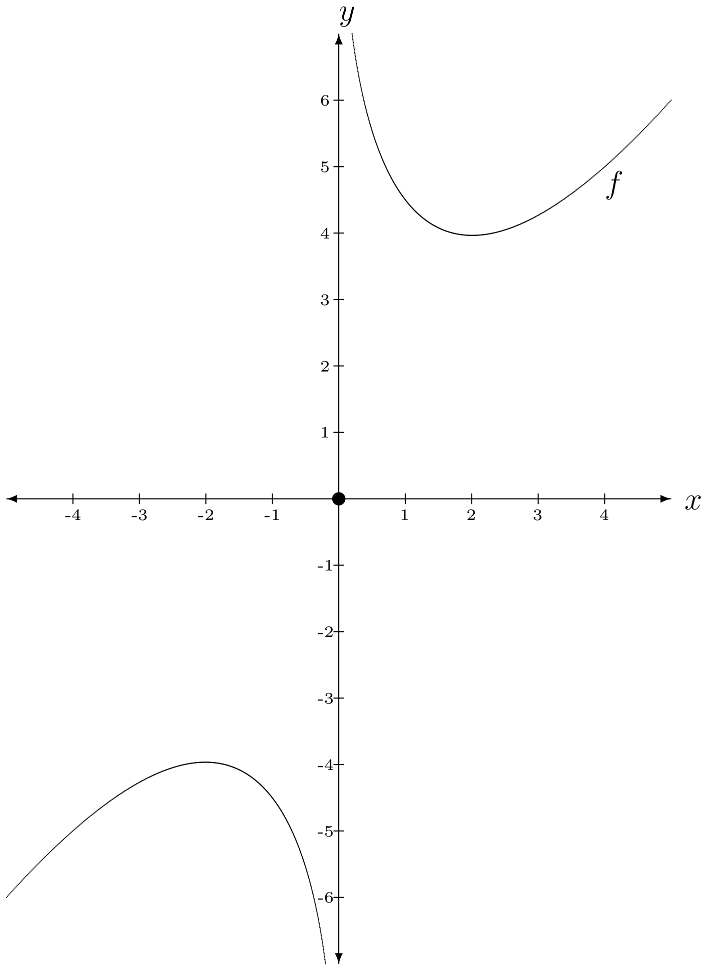
$$f''(-1) = \tfrac{8}{(-1)^3} = -8 < 0 \qquad f''(1) = \tfrac{8}{1^3} = 8 > 0.$$

Thus f is concave down on the interval $(-\infty, 0)$ and concave up on the interval $(0, \infty)$.

(d) From part (b) we see that f has a local maxima at -2 since -2 is in the domain of f and f is increasing on $(-\infty, -2)$ and decreasing on $(-2, 0)$. There is no local extrema at $x = 0$ since 0 is not in the domain of f. There is a local minima at $x = 2$ since f is decreasing on $(0, 2)$ and increasing on $(2, \infty)$.

f has no global maxima since f tends to arbitrarily large numbers as x tends to 0 from the right. f has no global minima since f tends to arbitrarily large negative numbers as x tends to 0 from the left.

(e)



7) The revenue is the number of mugs sold multiplied by the price for which the mugs are sold. Let x be the amount we charge per mug. When we are charging x dollars per mug we are going to sell $100 - 10(x - 20)$ mugs. If $R(x)$ is the revenue function then

$$R(x) = (100 - 10(x - 20))x.$$

The factor $(100 - 10(x - 20))$ represents the number of mugs sold when we are charging x dollars per mug and the factor of x represents the price for which each mug is sold. The goal is to optimize $R(x)$ for $x \geq 0$. We start by computing the derivative of $R(x)$. To compute $R'(x)$ we can use the Product Rule or we can simplify $R(x)$ and then use the Power Rule. We opt for the latter.

$$R(x) = (100 - 10(x - 20))x = (300 - 10x)x = -10x^2 + 300x.$$

Thus $R'(x) = -20x + 300$ and solving $R'(x) = 0$ gives $x = \frac{300}{20} = 15$. Since $R(x)$ is a parabola with a negative leading coefficient, $R(15)$ is the absolute maximum of $R(x)$ and we should charge 15 dollars per mug in order to maximize revenue.

8) (a) The function $y = 1 - e^x$ is continuous at 0 so $\lim_{x \to 0} 1 - e^x = 1 - e^0 = 1 - 1 = 0$. Similarly the function $y = \sin(3x)$ is continuous at 0 so $\lim_{x \to 0} \sin(3x) = \sin(3 \cdot 0) = \sin(0) =$

0. The limit is an indeterminate form and so we apply l'Hospital's Rule.

$$\begin{aligned}
\lim_{x\to 0}\frac{1-e^x}{\sin(3x)} &= \lim_{x\to 0}\frac{\frac{d}{dx}(1-e^x)}{\frac{d}{dx}(\sin(3x))} \\
&= \lim_{x\to 0}\frac{-e^x}{3\cos(3x)} \\
&= \frac{-e^0}{3\cos(3\cdot 0)} = \frac{-1}{3\cdot 1} = -\frac{1}{3}.
\end{aligned}$$

(b) Both of the functions $y = \ln x$ and $y = \cos x$ are continuous at π and $\cos\pi \neq 0$ so that

$$\lim_{x\to\pi}\frac{\ln x}{\cos x} = \frac{\ln\pi}{\cos\pi} = \frac{\ln\pi}{-1} = -\ln\pi.$$