(7/15/09)

Math 20B. Lecture Examples.

Section 10.5. The Ratio Test[†]

The ratio of the absolute values of successive terms in the geometric series $\sum_{n=0}^{\infty} r^n$ is $\frac{|r^{n+1}|}{|r^n|} = |r|$, and we know from Section 10.2 that the series converges absolutely if $|r| < 1$ and diverges if $|r| > 1$. This result has the following generalization.

Theorem 1 (The Ratio Test) **Suppose that $\mathbf{a_n}$ are nonzero numbers for $\mathbf{n \geq n_0}$ and that**

$$\lim_{\mathbf{n\to\infty}} \frac{|\,\mathbf{a_{n+1}}|}{|\,\mathbf{a_n}|} = \rho$$

where ρ (rho) is a nonnegative number or ∞.

(a) If $\rho < 1$, then $\sum_{\mathbf{j=n_0}}^{\infty} \mathbf{a_n}$ converges absolutely.

(b) If ρ is a number greater than 1 or is ∞, then $|\mathbf{a_n}|$ tends to ∞ as $\mathbf{n \to \infty}$ and the series diverges.

(c) If $\rho = 1$, the test gives no information about the series. It might converge absolutely, converge conditionally, or diverge.

If $\rho = 1$, as in part (c) of this theorem, we say that the Ratio Test fails or does not apply.

The Ratio Test is often used with series whose terms involve factorials, such as $n!$, or powers, such as a^n or n^n.

Example 1 **Apply the Ratio Test to $\sum_{\mathbf{n=0}}^{\infty} \frac{\mathbf{3^n}}{\mathbf{n!}}$.**

Answer: $\frac{|a_{n+1}|}{|a_n|} \to \rho = 0$ as $n \to \infty$ • The series converges absolutely because ρ is < 1. (Figure A1 shows a graph of the first 26 of its partial sums.)

$$y = \sum_{n=0}^{N} \frac{3^n}{n!}$$

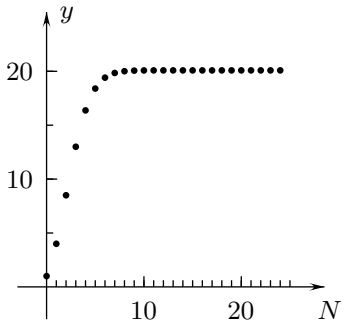


Figure A1

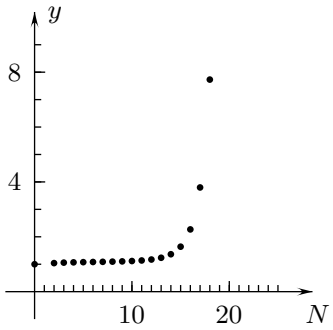
[†]Lecture notes to accompany Section 10.5 of *Calculus, Early Transcendentals* by Rogawski.

Example 2 **Apply the Ratio Test to** $\displaystyle\sum_{\mathbf{n=1}}^{\infty} \frac{\mathbf{n!}}{\mathbf{7^n}}$.

Answer: $\dfrac{|a_{n+1}|}{|a_n|} \to \rho = \infty$ as $n \to \infty$ • $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{7^n}$ diverges because $\rho > 1$. (Figure A2 shows a graph of the first 18 partial sums of the series.)

$$y = \sum_{n=0}^{N} \frac{n!}{7^n}$$

Figure A2



Example 3 **Apply the Ratio Test to (a)** $\displaystyle\sum_{\mathbf{n=1}}^{\infty} \frac{\mathbf{1}}{\mathbf{n^3}}$ **and (b)** $\displaystyle\sum_{\mathbf{n=1}}^{\infty} \frac{\mathbf{1}}{\mathbf{n^{1/2}}}$.

Answer: (a) The Ratio Test fails. (The series is, in fact, a convergent p-series.)
(b) The Ratio Test fails. (The series is, in fact, a divergent p-series.)

Example 4 **Does** $\displaystyle\sum_{\mathbf{n=1}}^{\infty} \frac{\mathbf{(-n)^3}}{\mathbf{4^n}}$ **converge absolutely, converge conditionally, or diverge?**

Answer: $\displaystyle\sum_{n=1}^{\infty} \frac{(-n)^3}{4^n}$ converges absolutely.

Example 5 **Does** $\displaystyle\sum_{\mathbf{n=0}}^{\infty} \frac{\mathbf{(-4)^n}}{\mathbf{n+1}}$ **converge absolutely, converge conditionally, or diverge?**

Answer: $\displaystyle\sum_{n=0}^{\infty} \frac{(-4)^n}{n+1}$ diverges.

Interactive Examples

Work the following Interactive Examples on Shenk's web page, http//www.math.ucsd.edu/~ashenk/:[‡]

Section 10.5: Examples 1, 2, 4, 5

[‡]The chapter and section numbers on Shenk's web site refer to his calculus manuscript and not to the chapters and sections of the textbook for the course.