

Math 20B. Lecture Examples.

Sections 10.6 and 10.7. Power series and Taylor series[†]

A POWER SERIES is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots .$$

Because the terms are constant multiples of powers of $x - c$, we say that this series is CENTERED at $x = c$.

Example 1 For what values of x does the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$$

converge?

Answer: The series converges for $-1 < x \leq 1$.

Graphs of partial sums $s_N = \sum_{n=1}^N \frac{(-1)^{n+1}}{n} x^n$ from Example 1 are shown in Figure 1 for $x = 0.75$,

where the series converges quickly, in Figure 2 for $x = 1$, where the convergence is relatively slow, and in Figure 3 for $x = 1.2$, where the series diverges.

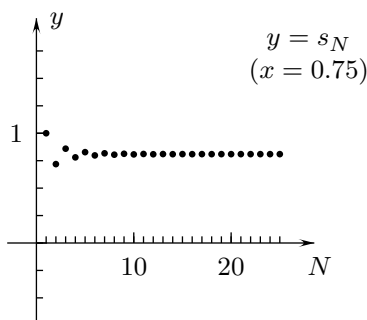


FIGURE 1

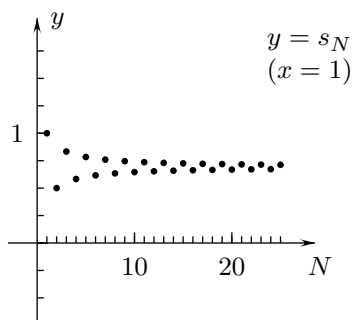


FIGURE 2

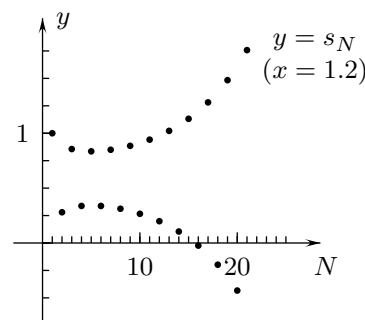


FIGURE 3

Theorem 1 For each power series $\sum_{n=0}^{\infty} a_n(x-c)^n$ there are three possibilities. Either

- (a) there is a positive number R such that the series converges absolutely for $|x - c| < R$ and diverges for $|x - c| > R$,
- (b) the series converges absolutely for all x , or
- (c) the series converges only for $x = c$.

The number R in part (a) of Theorem 1 is called the RADIUS OF CONVERGENCE of the series. In case (b), where the series converges for all x , we set $R = \infty$, and in case (c), where the series converges only at $x = c$, we set $R = 0$. The set of x 's where the series converges is called its INTERVAL OF CONVERGENCE. In Example 1 above the radius of convergence is $R = 1$ and the interval of convergence is $(-1, 1]$.

[†]Lecture notes to accompany Sections 10.6 and 10.7 of *Calculus, Early Transcendentals* by Rogawski.

If the radius of convergence R is a positive number, then the series may or may not converge at the endpoints $x = c \pm R$ of the interval of convergence. This is illustrated in Example 1, where the series converges at the right endpoint but not at the left endpoint of the interval $(-1, 1]$ of convergence.

Example 2 Find (a) the radius of convergence and (b) the interval of convergence of

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \left(\frac{x}{2} \right)^n.$$

Answer: (a) [Radius of convergence] = 2 (b) Interval of convergence: $[-2, 2]$

Example 3 What is the radius of convergence of $\sum_{n=0}^{\infty} \frac{1}{(2n)!} (x-1)^n$?

Answer: The radius of convergence is ∞ .

Example 4 What is the radius of convergence of $\sum_{n=1}^{\infty} n! x^n$?

Answer: The radius of convergence is 0.

Taylor and MacLaurin series

Recall from Section 8.4 that the N th-degree Taylor Polynomial approximation of $y = f(x)$ centered at c is

$$\begin{aligned} T_N(x) &= \sum_{n=0}^N \frac{1}{n!} f^{(n)}(c)(x-c)^n = f(c) + f'(c)(x-c) + \frac{1}{2}f''(c)(x-c)^2 \\ &\quad + \frac{1}{3!}f^{(3)}(c)(x-c)^3 + \cdots + \frac{1}{N!}f^{(N)}(c)(x-c)^N. \end{aligned}$$

The corresponding infinite series,

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(c)(x-c)^n &= f(c) + f'(c)(x-c) + \frac{1}{2}f''(c)(x-c)^2 \\ &\quad + \frac{1}{3!}f^{(3)}(c)(x-c)^3 + \frac{1}{4!}f^{(4)}(c)(x-c)^4 + \cdots \end{aligned}$$

is called the TAYLOR SERIES of f centered at c . The Taylor series centered at $c = 0$ is also called the MACLAURIN SERIES of f .

Example 5 (a) Give the MacLaurin series of $y = e^x$. (b) Find its radius of convergence.

Answer: (a) $y = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$ (b) $R = \infty$

The MacLaurin series for $y = e^x$ in Example 4 equals e^x , as in the first of the following formulas. Similar calculations give the second and third formulas, and the last is the geometric series.

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots \quad \text{for all } x \quad (2)$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots \quad \text{for all } x \quad (3)$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 + \cdots \quad \text{for all } x \quad (4)$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \quad \text{for } -1 < x < 1 \quad (5)$$

Example 6 Give the MacLaurin series of $f(x) = e^{-x}$.

$$\text{Answer: } e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n$$

Example 7 Find the MacLaurin series of $\frac{1}{1+t^2}$.

$$\text{Answer: } \frac{1}{1+t^2} = \sum_{n=0}^{\infty} (-1)^n t^{2n}$$

Theorem 3 (Differentiating and integrating MacLaurin series) Suppose that a MacLaurin series

$f(x) = \sum_{n=0}^{\infty} a_n x^n$ has a positive or infinite radius of convergence R . Then derivatives of $f(x)$ for x in $(-R, R)$ may be found by differentiating the power series term by term. Moreover, integrals $\int_{\alpha}^{\beta} f(x) dx$ of $f(x)$ with $-R < \alpha < \beta < R$ can be found by integrating the power series term by term.

Example 8 Use the differentiation formula $(1-x)^{-2} = \frac{d}{dx}[(1-x)^{-1}]$ to find the MacLaurin series for $f(x) = (1-x)^{-2}$.

$$\text{Answer: } (1-x)^{-2} = \sum_{n=1}^{\infty} nx^{n-1}$$

Example 9 Use the MacLaurin series $\frac{1}{1+t^2} = \sum_{n=0}^{\infty} (-1)^n t^{2n}$ for $|t| < 1$ from Example 7 to give the MacLaurin series of $y = \tan^{-1}x$.

$$\text{Answer: } \tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} x^{2n+1}$$

The complex exponential function $e^{i\theta}$ can be defined by using the MacLaurin series (2) for e^x with x replaced by $i\theta$:

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{1}{n!} (i\theta)^n = 1 + i\theta + \frac{1}{2}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \frac{1}{4!}(i\theta)^4 + \frac{1}{5!}(i\theta)^5 + \dots$$

This gives, with fomulas (3) and (4),

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{1}{2}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \frac{1}{4!}(i\theta)^4 + \frac{1}{5!}(i\theta)^5 + \dots \\ &= 1 + i\theta - \frac{1}{2}\theta^2 - \frac{1}{3!}i\theta^3 + \frac{1}{4!}\theta^4 + \frac{1}{5!}i\theta^5 + \dots \\ &= (1 - \frac{1}{2}\theta^2 - \frac{1}{4!}\theta^4 + \dots) + i(\theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 + \dots) \\ &= \cos \theta + i \sin \theta \end{aligned}$$

as stated in the course supplement on complex exponential functions.

Interactive Examples

Work the following Interactive Examples on Shenk's web page, <http://www.math.ucsd.edu/~ashenk/>:[‡]

Section 10.7: Examples 1a, 2a, 3, 5, and 6

[‡]The chapter and section numbers on Shenk's web site refer to his calculus manuscript and not to the chapters and sections of the textbook for the course.