Math 20C. Lecture Examples.

Section 14.8. Lagrange multipliers[†]

Imagine that the curve in Figure 1 is a mirror and that a viewer at point F_2 is looking at the image in the mirror of an object at point F_1. According to FERMAT'S PRINCIPLE from physics, the image will be the point P_0 on the mirror such that the total distance

$$f(P) = \overline{PF_1} + \overline{PF_2} \tag{1}$$

that the light travels from the object to the viewer is a minimum a that point. For each number c that is greater than the distance $\overline{F_1F_2}$ between the object and the viewer, the level curve $\overline{PF_1} + \overline{PF_2} = c$ of the distance $f(P)$ is an ellipse. Figure 2 shows eight of these ellipses.

Example 1 **Why can you expect the ellipse to be tangent to the mirror at the point $\mathbf{P_0}$?**

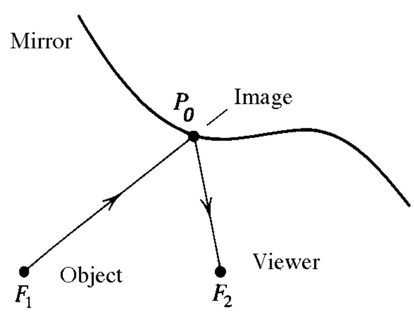


FIGURE 1

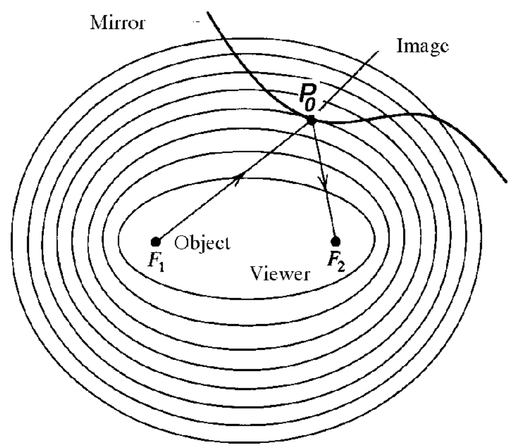


FIGURE 2

Answer: The solution is the answer

To express the result of Example 1 with formulas, we introduce xy-axes, as in Figure 3, and let $f(x, y)$ be the sum **(1)** of the distances from $P = (x, y)$ to F_1 and F_2. We also assume that the mirror C is a level curve $g(x, y) = c$ of another function with a nonzero gradient vector.

[†]Lecture notes to accompany Section 14.8 of *Calculus, Early Transcendentals* by Rogawski.

At $P_0 = (a, b)$ where the smallest ellipse and C are tangent, $\nabla f(a, b)$ is perpendicular to the ellipse, which is the level curve of f, and $\nabla g(a, b)$ is perpendicular to C, which is the level curve of g (Figure 3). Since the curves are tangent at (a, b), the two gradient vectors are parallel, and there is a number λ such that $\nabla f(a, b) = \lambda \nabla g(a, b)$. The number λ is called a LAGRANGE MULTIPLIER.

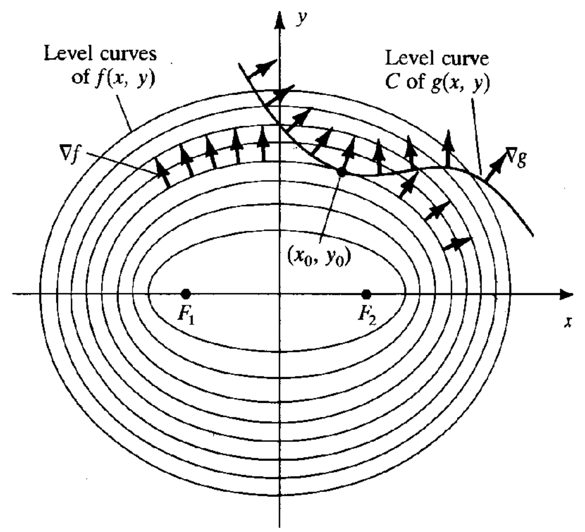


FIGURE 3

Example 2 **Use Lagrange multipliers to find the maximum and minimum values of $f(x, y) = x - 2y + 1$ on the ellipse $x^2 + 3y^2 = 21$ and where they occur.**

Answer: The maximum of f on the ellipse is 8 at $(3, -2)$ and the minimum is -6 at $(-3, 2)$.

Example 3 **Sketch the ellipse in Example 2 and the level curves of $z = f(x, y)$ where the maximum and minimum occur.**

Answer: Figure A3

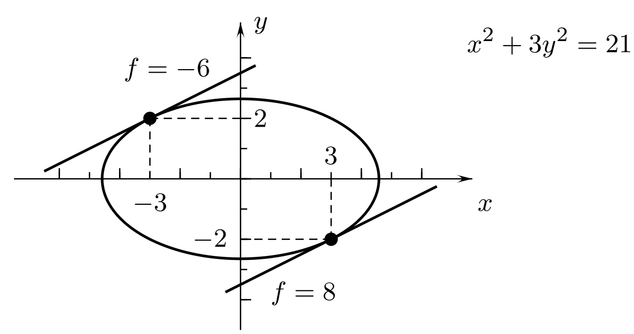


Figure A3

Example 4 **Use Lagrange multipliers to find the rectangle of perimeter 12 that has the largest area (Figure 4).**

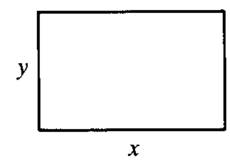


FIGURE 4

Answer: The rectangle with perimeter 12 and maximum area is a square of width 3 and area 9. • (The level curves of the area $A = xy$ are the hyperbolas $xy = k$ and the constraint curve is the line $2x + 2y = 12$ in Figure A5.)

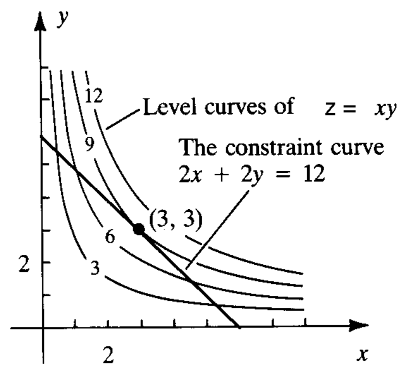


Figure A5

Example 5 **Use Lagrange multipliers to find the dimensions of the rectangle of area 9 that has the shortest perimeter.**

Answer: The rectangle of area 9 and minimum perimeter is the square of width 3, and perimeter 12. • (The level curves of the perimeter $P = 2x + 2y$ are the lines $2x + 2y = c$ and the constraint curve is the hyperbola $xy = 9$ in Figure A6.

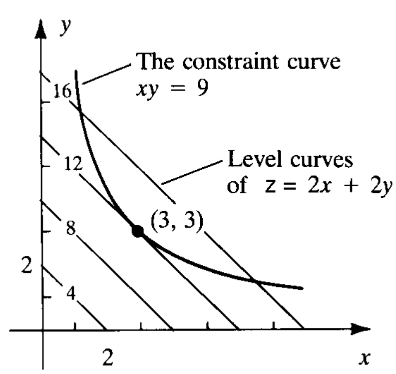


Figure A5

Example 6 **Solve Example 2 by finding the minimum of a function of one variable.**

Answer: The constraint condition $2x + 2y = 12$ implies that $y = 6 - x$, which in the formula $A = xy$ for the area gives $A(x) = x(6 - x) = 6x - x^2$. • $y = A(x)$ has a global maximum for $x = 3$, where $y = 6 - x = 3$.

Example 7 **Figure 5 shows the curve C: $\mathbf{x^2y = 8}$ and level curves of the function $\mathbf{f(x,y) = \frac{1}{2}x^2 + y}$. Find the minimum value of f on C and where it occurs.**

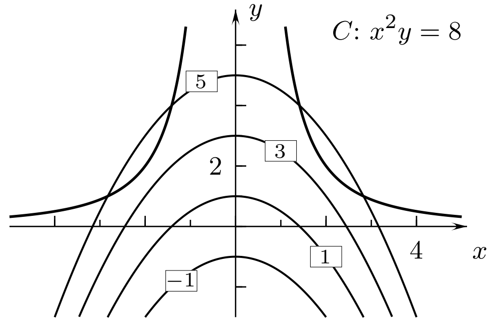


FIGURE 5

Answer: Set $g = x^2y$. • $\nabla f = \lambda \nabla g$ • $\lambda = \pm\frac{1}{4}$ • [Minimum] = 4 at $(\pm 2, 2)$ • Figure A7

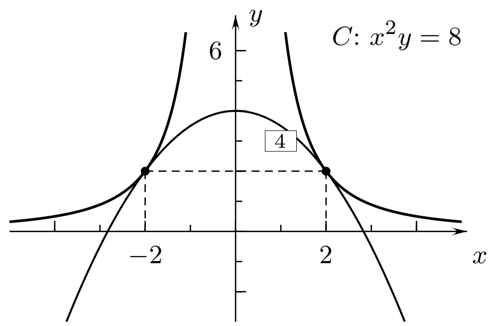
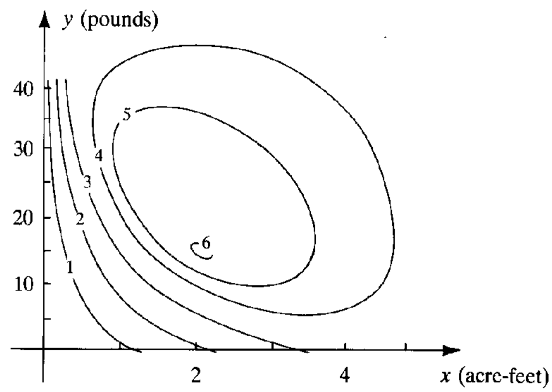


Figure A7

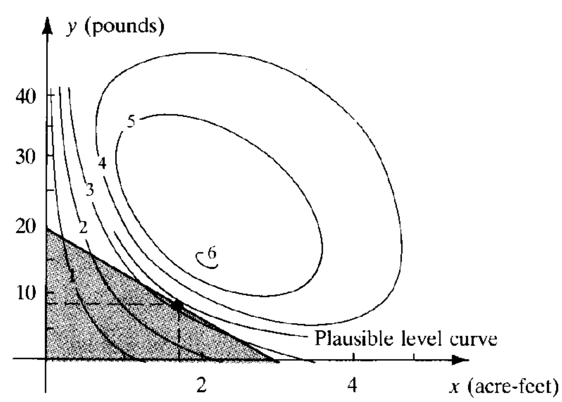
Example 8 **Figure 6 shows level curves of the yield of corn, measured in thousand of pounds per acre, that a farmer will obtain if he applies x acre-feet of irrigation water and y pounds of fertilizer per acre during the growing season.**[†] **Suppose that the water costs \$60 per acre-foot, the fertilizer costs 9 dollars per pound, and the farmer has \$180 to invest per acre for water and fertilizer. Approximately how much water and how much fertilizer should he buy to maximize the yield of corn?**

FIGURE 6



Answer: It would cost $C(x, y) = 60x + 9y$ dollars for x acre-feet of water and y-pounds of fertilizer. • $x \geq 0, y \geq 0, 60x + 9y \leq 180$ and consequently (x, y) must be in the shaded triangle of Figure A8. • The farmer should use approximately 1.75 acre-feet of water and 8.3 pounds of fertilizer for each acre.

Figure A8



Example 9 **Find the maximum and minimum values of $\mathbf{f(x, y, z) = 6x + 3y + 2z - 5}$ on the ellipsoid $\mathbf{4x^2 + 2y^2 + z^2 = 70}$.**

Answer: Set $g = 4x^2 + 2y^2 + z^2$. • $\nabla f = \lambda \nabla g$ • $\lambda = \pm\frac{1}{4}$ • The maximum is 30 and the minimum is -40.

Interactive Examples

Work the following Interactive Examples on Shenk's web page, http//www.math.ucsd.edu/˜ashenk/:[‡]

Section 15.3: Examples 1–3

[†] An acre foot of water would cover an acre one-foot deep.

[‡] The chapter and section numbers on Shenk's web site refer to his calculus manuscript and not to the chapters and sections of the textbook for the course.