Math 20C. Lecture Examples.

Section 14.1, Part 1. Functions of two variables†

Example 1 **(a) What is the domain of $f(x,y) = x^2 + y^2$? (b) What are the values $f(2,3)$ and $f(-2,-3)$ of this function at (2,3) and $(-2,-3)$? (c) What is its range?**

Answer: **(a)** The domain of f is the entire xy-plane. **(b)** $f(2,3) = 13$ • $f(-2,-3) = 13$. **(c)** The range of f is the closed infinite interval $[0, \infty)$.

Example 2 **Determine the shape of the surface $z = x^2 + y^2$ in xyz-space by studying its cross sections in the planes $x = c$ perpendicular to the x-axis.**

Answer: The intersection of the surface $z = x^2 + y^2$ with the plane $x = c$ is a parabola that opens upward and whose vertex is at the origin if $c = 0$ and is c^2 units above the xy-plane if $c \neq 0$ • Figure A2a • The surface has the bowl-like shape in Figure A2b

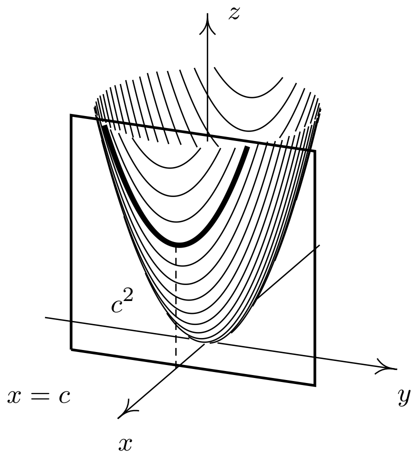


Figure A2a

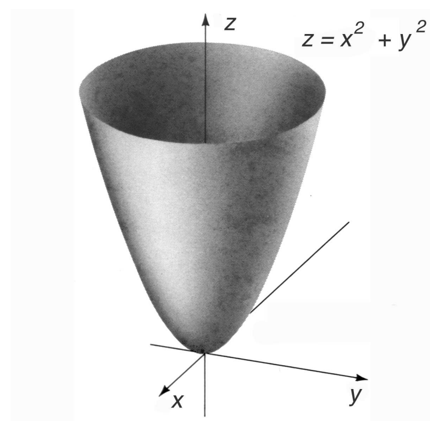


Figure A2b

† Lecture notes to accompany Section 14.1, Part 1 of *Calculus, Early Transcendentals* by Rogawski.

Example 3 **Determine the shape of the surface $z = x^2 + y^2$ of Example 2 by studying its cross sections in the planes $y = c$ perpendicular to the y-axis.**

Answer: The intersection of the surface $z = x^2 + y^2$ with the plane $y = c$ is parabola that opens upward and whose vertex is at the origin if $c = 0$ and is c^2 units above the xy-plane if $c \neq 0$. • Figure A3a • The surface has the bowl-like shape from Example 2. (Figure A3b shows the cross sections from Examples 2 and 3 together.)

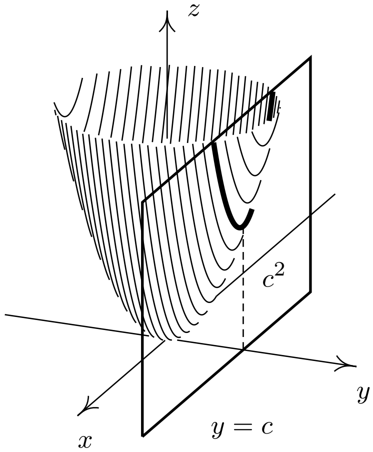


Figure A3a

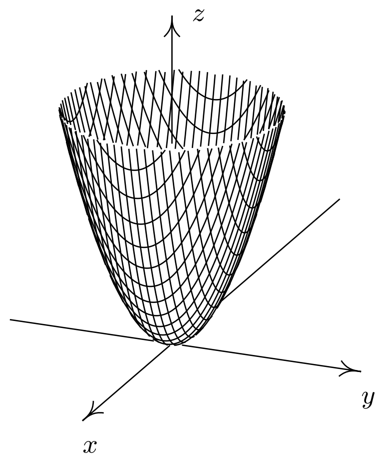


Figure A3b

Example 4 **Determine the shape of the surface $z = y^2 - x^2$ by studying its cross sections in the planes $x = c$ perpendicular to the x-axis.**

Answer: The intersection of the surface $z = y^2 - x^2$ with the plane $x = c$ is a parabola that opens upward and whose vertex is c^2 units below the xz-plane. • Figure A4a • The vertex is at the origin for $c = 0$ and drops below the xy-plane as c moves away from zero. • The surface has the saddle shape in Figure A4b.

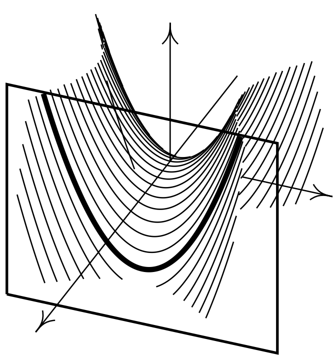


Figure A4a

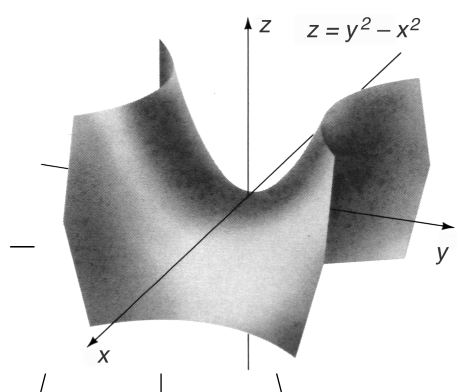


Figure A4b

Example 5 **Determine the shape of the surface $\mathbf{z = y^2 - x^2}$ of Example 4 by studying its cross sections in the planes $\mathbf{y = c}$ perpendicular to the $\mathbf{y}$-axis.**

Answer: The intersection of the surface $z = y^2 - x^2$ with the plane $y = c$ is a parabola that opens downward and whose vertex is c^2 units above the xy-plane. • Figure A5a • The vertex is at the origin for $c = 0$ and rises above the xy-plane as c moves away from zero. • The surface has the saddle shape from Example 4. (Figure A5b shows the two sets of cross sections together.)

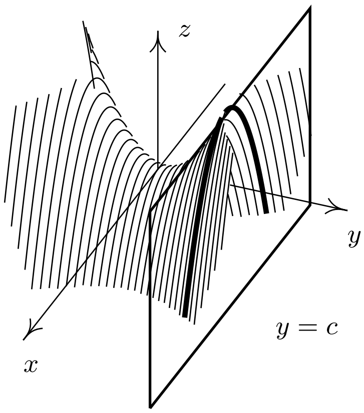


Figure A5a

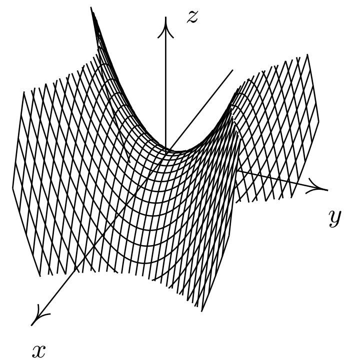


Figure A5b

Example 6 **Use the curve $\mathbf{z = y - \frac{1}{12}y^3}$ in the yz-plane of Figure 1 to determine the shape of the surface $\mathbf{z = y - \frac{1}{12}y^3 - \frac{1}{4}x^2}$.**

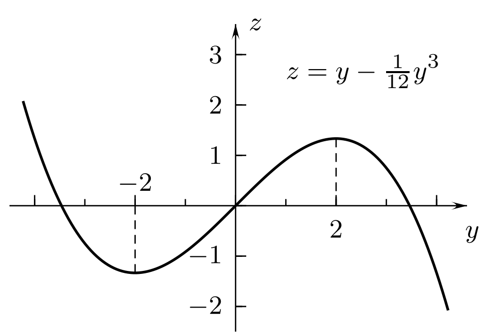


FIGURE 1

Answer: One solution: The cross section of the surface in the plane $x = c$ has the shape of the curve in Figure 1 if $c = 0$, is that curve moved down and forward if $c > 0$ and is that curve moved down and back if $c < 0$. •
The surface has the boot-like shape in Figure A6
Another solution: The cross section in the plane $y = c$ is a parabola that opens downward and has its vertex on the curve in Figure 1. • The surface has the boot-like shape in Figure A6.

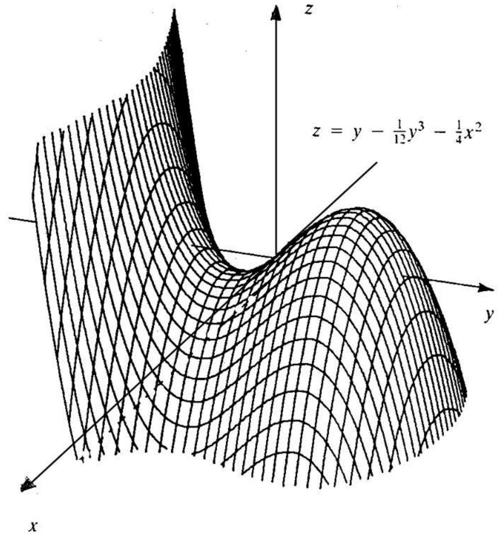


Figure A6

Example 7 **Describe the level curves of the function $\mathbf{f(x,y) = x^2 + y^2}$ from Examples 2 and 3.**

Answer: Figure A7a shows horizontal cross sections of the graph of f and Figure A7b shows the corresponding level curves. • The level curve $f = c$ is the circle of radius $\sqrt{c}$ with its center at the origin if $c > 0$, is the origin if $x = 0$, and is empty if $c < 0$. (The surface is called a "circular paraboloid.")

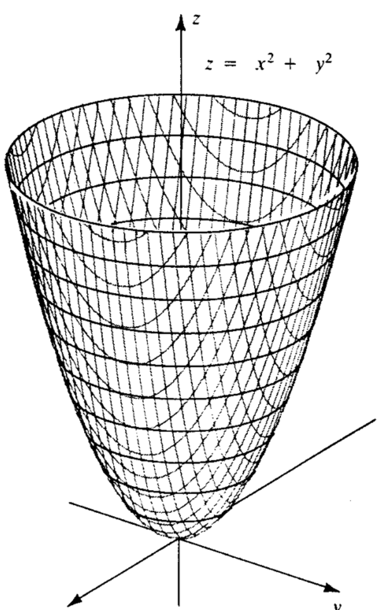


Figure A7a

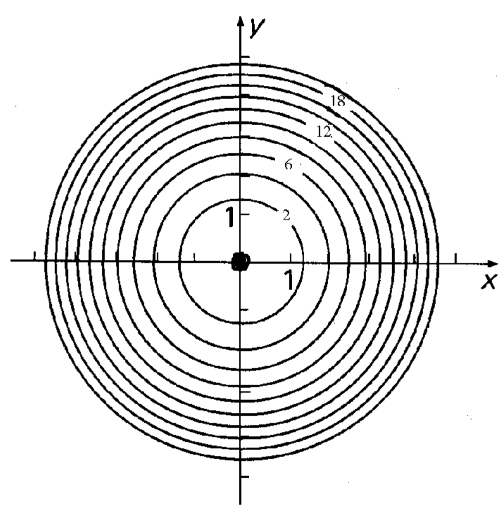


Figure A7b

Example 8 **Describe the level curves of $\mathbf{g(x,y) = y^2 - x^2}$ from Examples 4 and 5.**

Answer: Figures A8a and A8b • The level curves $g = c$ is a hyperbola with the equation $y^2 - x^2 = c$. (The surface is a "hyperbolic paraboloid.")

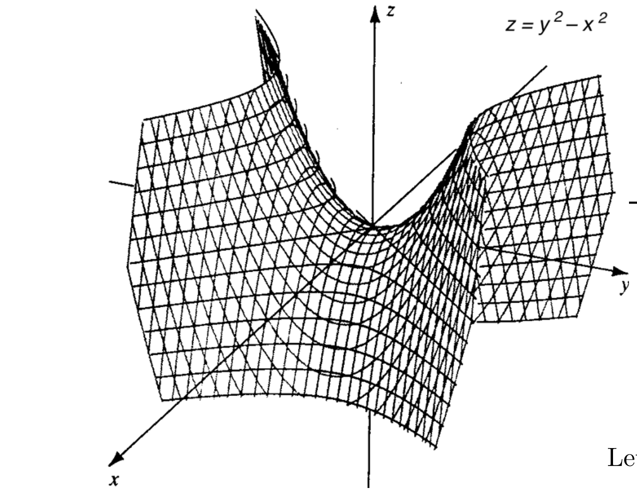
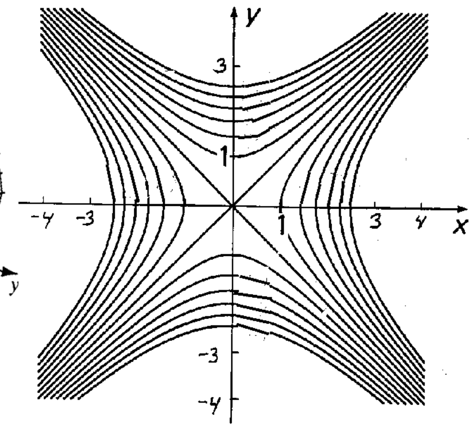


Figure A8a



Level curves of $g(x, y) = y^2 - x^2$

Figure A8b

Example 9 **Figures 2 and 3 show horizontal cross sections of the graph of $\mathbf{h(x, y) = y - \frac{1}{12}y^3 - \frac{1}{4}x^2}$ from Example 6 and the corresponding level curves of the function. Describe how the surface can be reconstructed from the level curves.**

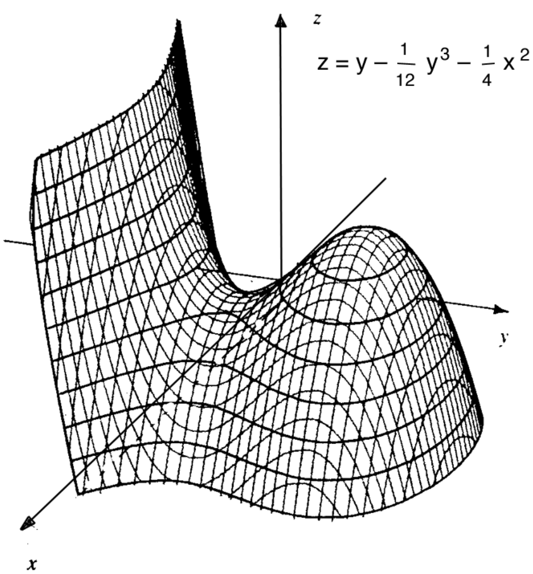


FIGURE 2

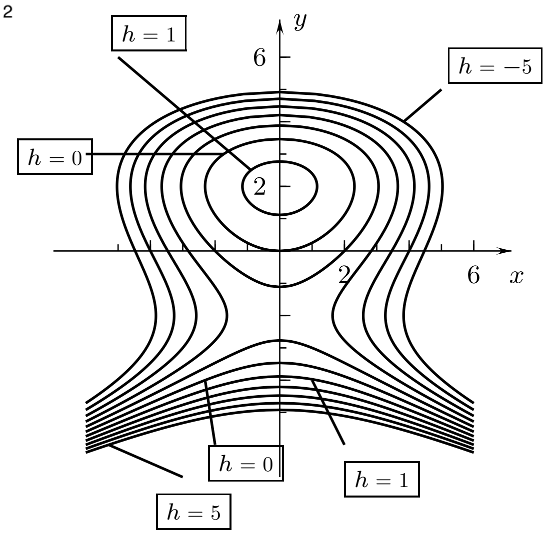


FIGURE 3

Answer: Leave the two parts of the level curve $h = 0$ on the xyplane. • Raise the two parts of the curve labeled $h = 1$ one unit on the "toe" and "leg" of the "boot." • Lower the curves above the upper part of $h = 1$ to form the sides of the "boot." • Raise the curves below the lower part of $h = 1$ to form the more of the "leg" of the "boot."

Rotating axes

You will see in Section 14.8 that the surfaces $z = k\,xy$ with nonzero constants k are important in the study of maxima and minima of functions with two variables. Their shapes can be determined by introducing new $x'y'$-coordinates by rotating the x- and y-axes 45° counterclockwise as in Figure 4.† The original coordinates (x, y) can be calculated from the new coordinates (x', y') by the formulas,

$$\mathbf{x} = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}(\mathbf{x}' - \mathbf{y}'),\ \mathbf{y} = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}(\mathbf{x}' + \mathbf{y}'). \tag{1}$$

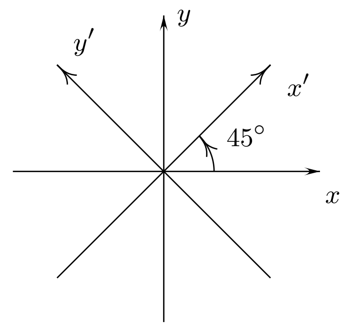


FIGURE 4

Example 10 **Use $\mathbf{x'y'}$-coordinates as in Figure 4 to analyze the surface $\mathbf{z = -2xy}$.**

Answer: The graph is the surface $z = (y')^2 - (x')^2$, so it is the surface of Figure A4b rotated 45° as in Figure A10. (Notice that the x- and y-axes are on the surface.)

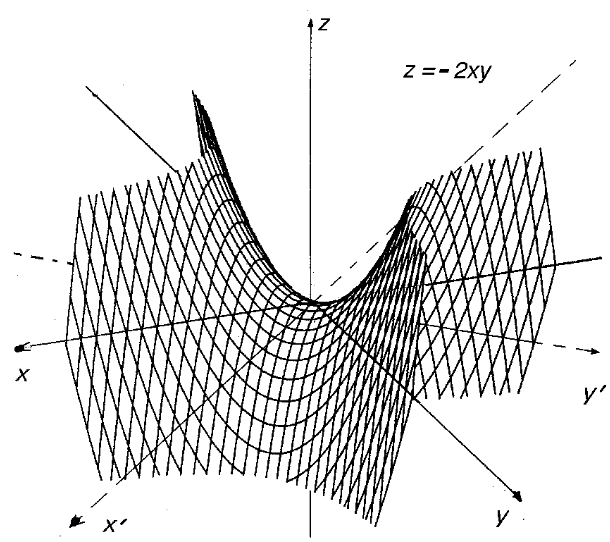


Figure A10

Interactive Examples

Work the following Interactive Examples on Shenk's web page, http//www.math.ucsd.edu/˜ashenk/:‡

Section 14.1: Examples 1–6

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† The primes on the variables x' and y' here are just to distinguish them from x and y. They do not denote derivatives.

‡ The chapter and section numbers on Shenk's web site refer to his calculus manuscript and not to the chapters and sections of the textbook for the course.