(10/1/08)

Math 20C. Lecture Examples.

Section 14.5. Directional derivatives and gradient vectors†

Example 1 **(a) Find the directional derivative of $f(x,y) = x^2 + y^2$ at $(1,-1)$ in the direction of the unit vector u= $\langle \frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2}\rangle$ (Figure 1).**
(b) Why is it plausible that the directional derivative is positive?

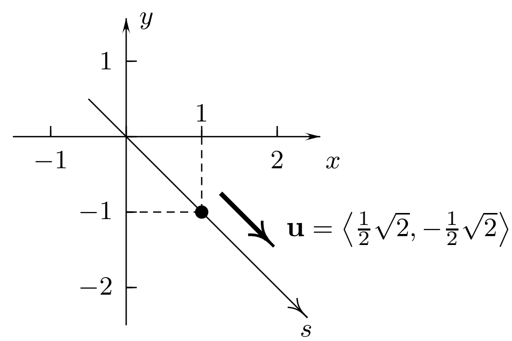


FIGURE 1

Answer: (a) $D_{\mathbf{u}}f(1,-1) = 2\sqrt{2}$ **(b)** $f(x,y) = x^2 + y^2$ is increasing in the direction of **u** at $(1,-1)$ in Figure 1 because its graph is a circular paraboloid that opens upward.

Example 2 **What is the derivative of $f(x,y) = x^2y^5$ at $P = (2,1)$ in the direction toward $Q = (4,0)$?**

Answer: $D_{\mathbf{u}}f(2,1) = -2\sqrt{5}$

Example 3 **What is the derivative of $h(x,y) = e^{xy}$ at (2,3) in the direction at an angle of $\frac{2}{3}\pi$ radians from the positive x-direction?**

Answer: Figure A3 • $\mathbf{u} = \langle -\frac{1}{2}, \frac{1}{2}\sqrt{3}\rangle$ • $D_{\mathbf{u}}h(2,3) = (-\frac{3}{2} + \sqrt{3})e^6$

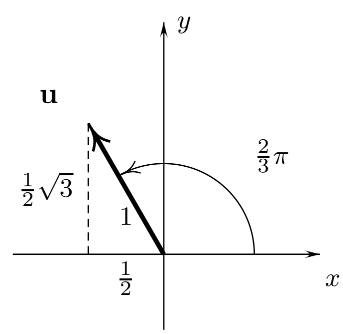


Figure A3

†Lecture notes to accompany Section 14.5 of *Calculus, Early Transcendentals* by Rogawski.

Example 4 **Figure 2 shows level curves of the temperature $\mathbf{T} = \mathbf{T}(\mathbf{x}, \mathbf{y})$ (degrees Celsius) of the surface of the ocean off the west coast of the United States at one time.[(1)] Find the approximate rate of change of the temperature toward the northeast at point $\mathbf{P}$ in the drawing.**

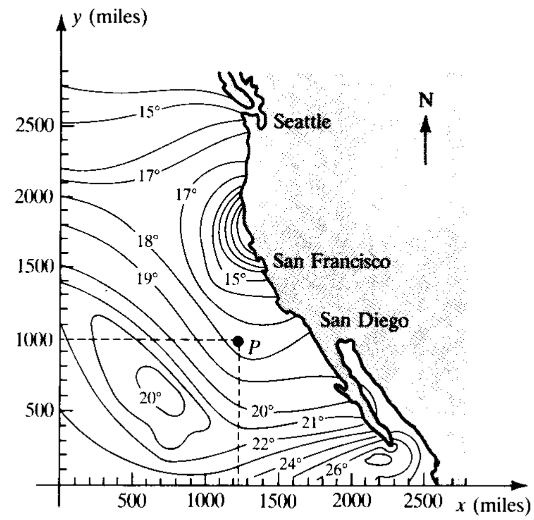


FIGURE 2

Answer: One answer: Figure A4 • $D_{\mathbf{u}}T(P) \approx -0.005$ degrees per mile

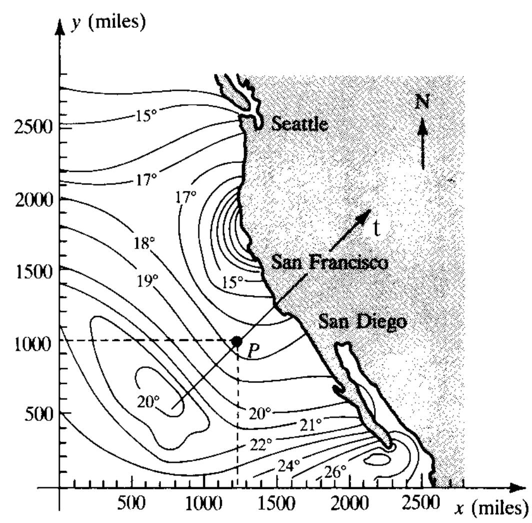


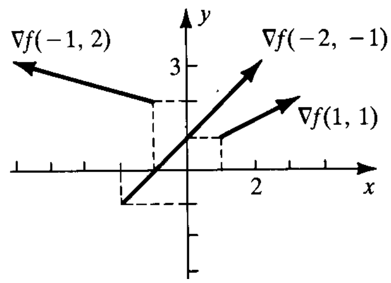
Figure A4

(1) Data adapted from *Zoogeography of the Sea* by S. Elkman, London: Sidgwich and Jackson, 1953, p. 144.

Example 5 **Draw $\nabla f(1,1), \nabla f(-1,2)$, and $\nabla f(-2,-1)$ for $f(x,y) = x^2y$. Use the scale on the x- and y-axes to measure the lengths of the arrows.**

Answer: $\nabla f(1,1) = \langle 2,1\rangle$ • $\nabla f(-1,2) = \langle -4,1\rangle$ • $\nabla f(-2,-1) = \langle 4,4\rangle$ • Figure A5

Figure A5



Example 6 **(a) What is the maximum directional derivative of $g(x,y) = y^2e^{2x}$ at $(2,-1)$ and in the direction of what unit vector does it occur?**
(b) What is the minimum directional derivative of g at $(2,-1)$ and in the direction of what unit vector does it occur?

Answer: (a) The maximum directional derivative is $\sqrt{8}\,e^4$ and occurs in the direction of $\mathbf{u} = \dfrac{\langle 1,-1\rangle}{\sqrt{2}}$.

(b) The minimum directional derivative is $= -\sqrt{8}\,e^4$ and occurs in the direction of $\mathbf{u} = \dfrac{\langle -1,1\rangle}{\sqrt{2}}$.

Example 7 **Give the two unit vectors u such that the function $z = g(x,y)$ of Example 6 has zero derivatives at $(2,-1)$ in the direction of u.**

Answer: The directional derivative is zero in the directions of $\mathbf{u} = \dfrac{\langle -1,-1\rangle}{\sqrt{2}}$ and $\mathbf{u} = \dfrac{\langle 1,1\rangle}{\sqrt{2}}$.

Example 8 **(a) Draw the gradient vector of $f(x,y) = xy$ at (1,2) and the level curve of f through that point.**
(b) Draw $\nabla f(-3,1)$ and the level curve of f through $(-3,1)$. Use the scales on the axes to measure the components.

Answer: (a) $\nabla f(1,2) = \langle 2,1\rangle$ • The level curve is $y = \dfrac{2}{x}$ • Figure A8a

(b) $\nabla f(-3,1) = \langle 1,-3\rangle$ • The level curve is $y = \dfrac{-3}{x}$. • Figure A8b

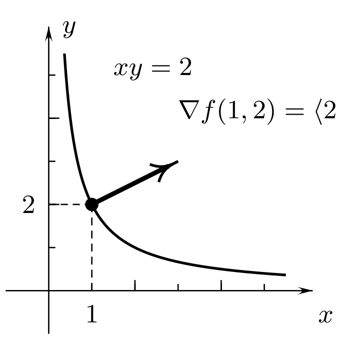


Figure A8a

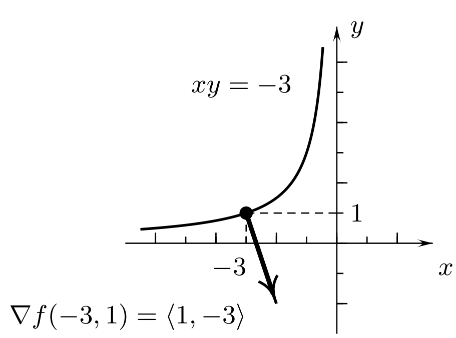


Figure A8b

Example 9 **(a) What is the gradient of $\mathbf{f(x,y,z) = xyz}$ at (1,2,3)?**
(b) What is the directional derivative of f at (1,2,3) in the direction toward (2,3,4)?
(c) What is the greatest directional derivative of f at (1,2,3)?

Answer: (a) $\nabla f(1,2,3) = \langle 6,3,2\rangle$ **(b)** The directional derivative of f at $(1,2,3)$ in the direction toward $(2,3,4)$ is $\frac{11}{\sqrt{3}}$. **(c)** The greatest directional derivative of f at $(1,2,3)$ is 7.

Example 10 **Give an equation of the tangent plane at the point (2, 2,3) on the hyperboloid of two sheets $\mathbf{x^2 + y^2 - z^2 = -1}$.**

Answer: Tangent plane: $4(x-2)+4(y-2)-6(z-3)=0$ or $2x+2y-3z=-1$

Interactive Examples

Work the following Interactive Examples on Shenk's web page, http//www.math.ucsd.edu/˜ashenk/:[‡]

Section 14.5: Examples 1 through 6

Section 14.7: Examples 4, 5, and 6

[‡]The chapter and section numbers on Shenk's web site refer to his calculus manuscript and not to the chapters and sections of the textbook for the course.