(10/1/08)

Math 20C. Lecture Examples.

Section 14.7, Part 2. Maxima and minima: The Second-Derivative Test†

The second-degree Taylor polynomial approximation of $y = f(x)$ at $x = a$ is

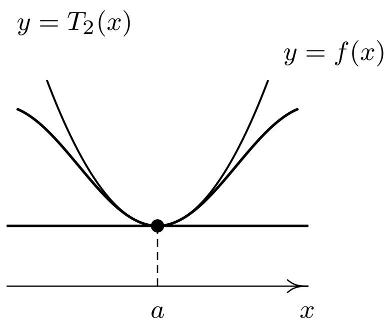
$$T_2(x) = f(a) + f'(a)(x-a) + \tfrac{1}{2}f''(a)(x-a)^2.$$

This polynomial has the same value and first two derivatives as $y = f(x)$ at a and, consequently, is the second-degree polynomial that best approximates $y = f(x)$ near a.

If a is a critical point of $y = f(x)$, then $f'(a) = 0$ and the Taylor polynomial is

$$T_2(x) = f(a) + \tfrac{1}{2}f''(a)(x-a)^2.$$

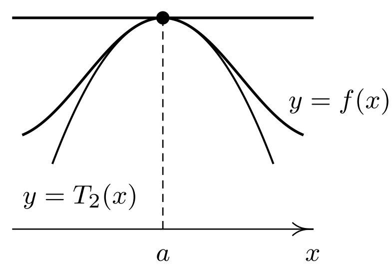
If $f''(a)$ is positive, then the graph of T_2 is a parabola that opens upward, as in Figure 1, and f has a local minimum at a. If $f''(a)$ is negative, then the graph of T_2 is a parabola that opens downward, as in Figure 4, and f has a local maximum at a.



$y = T_2(x)$ opens upward

Local minimum

FIGURE 1



$y = T_2(x)$ opens downward

Local maximum

FIGURE 2

To study local maxima and minima in the case of two variables, we approximate functions $z = f(x,y)$ by second-degree Taylor polynomials.

Definition 1 *The second-degree Taylor approximation of $z = f(x,y)$ at (a,b) is*

$$T_2(x,y) = D + E(x-a) + F(y-b) + \tfrac{1}{2}A(x-a)^2 + B(x-a)(y-b) + \tfrac{1}{2}C(y-b)^2 \quad \textbf{(1)}$$

where $D = f(a,b)$ is the value of f, and $E = f_x(a,b), F = f_y(a,b), A = f_{xx}(a,b)$, $B = f_{xy}(a,b)$, and $C = f_{yy}(a,b)$ are its first- and second-order derivatives at (a,b).

The Taylor polynomial **(1)** has the same value and the same first and second derivatives as f at (a,b) and is the second-degree polynomial that best approximates f near that point.

†Lecture notes to accompany Section 14.7, Part 2 of *Calculus, Early Transcendentals* by Rogawski.

Example 1 **Give the second-degree Taylor polynomial approximation of $\mathbf{f(x,y) = 1 - \cos x \cos y}$ at $\mathbf{x = 0, y = 0}$.**

Answer: $T_2(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$

The graph of $f(x,y) = 1 - \cos x \cos y$ from Example 1 is in Figure 3. The graph of its Taylor polynomial approximation is the circular paraboloid $z = T_2(x,y)$ shown in Figure 4. The function f has a local minimum at $x = 0, y = 0$ because the Taylor polynomial has a global minimum there.

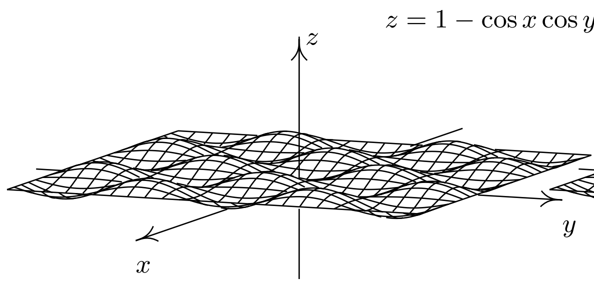


FIGURE 3

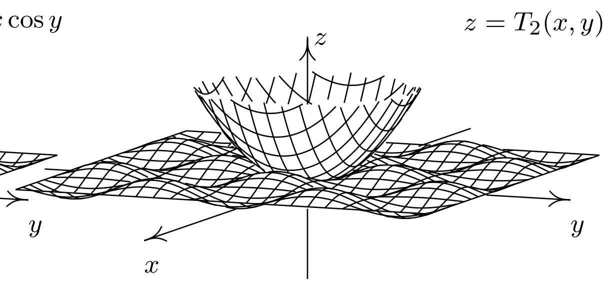
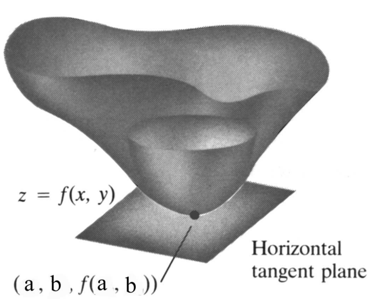


FIGURE 4

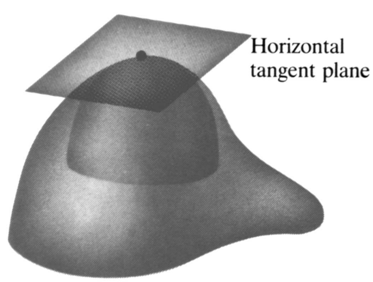
This approach can be applied to any function. Suppose that (a,b) is a critical point of f. Then $f_x(a,b) = 0$ and $f_y(a,b) = 0$ and the Taylor polynomial approximation of f at (a,b) **(1)** is

$$T_2(x,y) = D + \tfrac{1}{2}A(x-a)^2 + B(x-a)(y-b) + \tfrac{1}{2}C(y-b)^2. \tag{2}$$

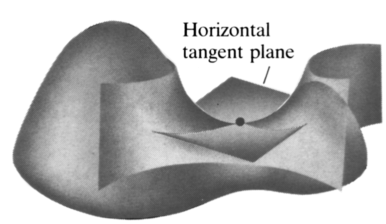
If the graph of T_2 is an elliptic paraboloid that opens upward as in Figure 5, then f has a local minimum at $x = a, y = b$; if the graph of T_2 is an elliptic paraboloid that opens downward as in Figure 6, then f has a local maximum at $x = a, y = b$; and if the graph of T_2 is a hyperbolic paraboloid as in Figure 7, then f has a SADDLE POINT, which is neither a local maximum nor local minimum, at $x = a, y = b$. These geometric ideas are the basis of the Second-Derivative Test with two variables.



Local minimum
FIGURE 5



Local maximum
FIGURE 6



Saddle point
FIGURE 7

Example 2 **Figures 8 and 9 show the graph of $\mathbf{f = -x^4 - y^4 - 4xy + \frac{1}{16}}$ and its level curves. Use the Second-Derivative Test to classify its critical points.**

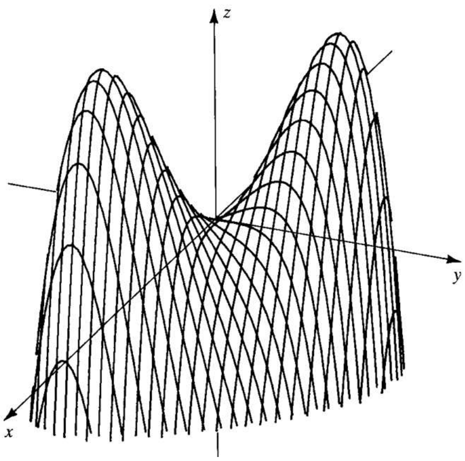


FIGURE 8

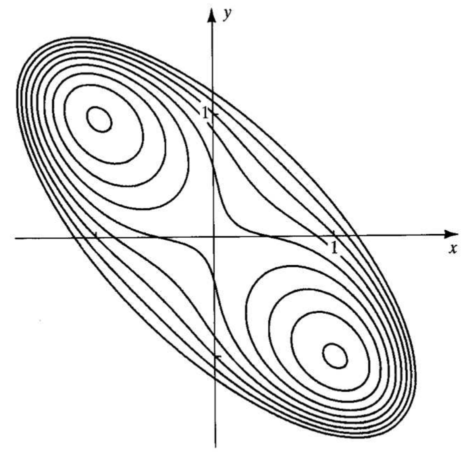


FIGURE 9

Answer: f has a saddle point at $(0,0)$ and local maxima at $(1,-1)$ and at $(-1,1)$.

Example 3 **Figures 10 and 11 show the graph of $\mathbf{f = -2x^3 - 3y^4 + 6xy^2}$ and its level curves. Use the Second-Derivative Test to classify its critical points.**

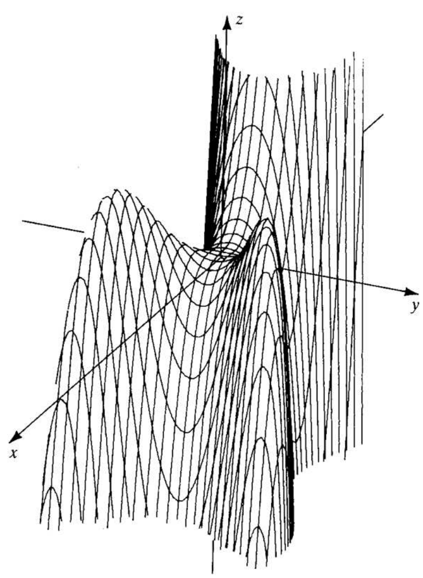


FIGURE 10

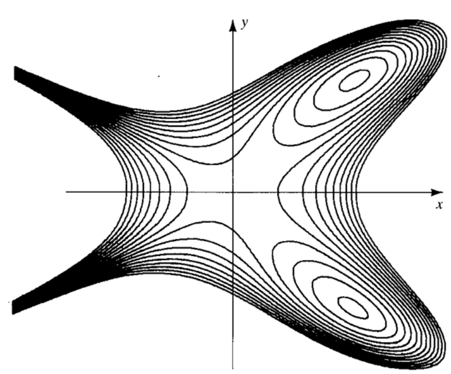


FIGURE 11

Answer: The function has local maxima at $(1,1)$ and $(1,-1)$. • The Second-Derivative Test fails at $(0,0)$.

Interactive Examples

Work the following Interactive Examples on Shenk's web page, http//www.math.ucsd.edu/˜ashenk/:[‡]

Section 15.2: Examples 1–3

[‡]The chapter and section numbers on Shenk's web site refer to his calculus manuscript and not to the chapters and sections of the textbook for the course.