

Math 20C. Lecture Examples.

Section 14.7, Part 2. Maxima and minima: The Second-Derivative Test[†]

The second-degree Taylor polynomial approximation of $y = f(x)$ at $x = a$ is

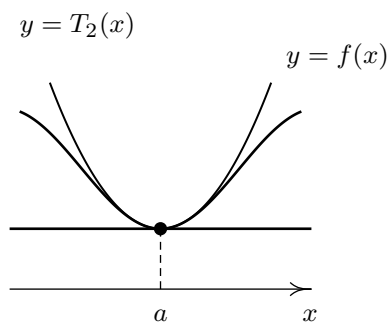
$$T_2(x) = f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2.$$

This polynomial has the same value and first two derivatives as $y = f(x)$ at a and, consequently, is the second-degree polynomial that best approximates $y = f(x)$ near a .

If a is a critical point of $y = f(x)$, then $f'(a) = 0$ and the Taylor polynomial is

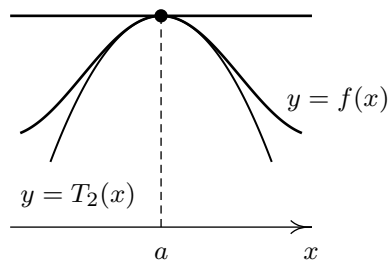
$$T_2(x) = f(a) + \frac{1}{2}f''(a)(x - a)^2.$$

If $f''(a)$ is positive, then the graph of T_2 is a parabola that opens upward, as in Figure 1, and f has a local minimum at a . If $f''(a)$ is negative, then the graph of T_2 is a parabola that opens downward, as in Figure 4, and f has a local maximum at a .



$y = T_2(x)$ opens upward
Local minimum

FIGURE 1



$y = T_2(x)$ opens downward
Local maximum

FIGURE 2

To study local maxima and minima in the case of two variables, we approximate functions $z = f(x, y)$ by second-degree Taylor polynomials.

Definition 1 The second-degree Taylor approximation of $z = f(x, y)$ at (a, b) is

$$T_2(x, y) = D + E(x - a) + F(y - b) + \frac{1}{2}A(x - a)^2 + B(x - a)(y - b) + \frac{1}{2}C(y - b)^2 \quad (1)$$

where $D = f(a, b)$ is the value of f , and $E = f_x(a, b)$, $F = f_y(a, b)$, $A = f_{xx}(a, b)$, $B = f_{xy}(a, b)$, and $C = f_{yy}(a, b)$ are its first- and second-order derivatives at (a, b) .

The Taylor polynomial (1) has the same value and the same first and second derivatives as f at (a, b) and is the second-degree polynomial that best approximates f near that point.

[†]Lecture notes to accompany Section 14.7, Part 2 of *Calculus, Early Transcendentals* by Rogawski.

Example 1 Give the second-degree Taylor polynomial approximation of $f(x, y) = 1 - \cos x \cos y$ at $x = 0, y = 0$.

Answer: $T_2(x, y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$

The graph of $f(x, y) = 1 - \cos x \cos y$ from Example 1 is in Figure 3. The graph of its Taylor polynomial approximation is the circular paraboloid $z = T_2(x, y)$ shown in Figure 4. The function f has a local minimum at $x = 0, y = 0$ because the Taylor polynomial has a global minimum there.

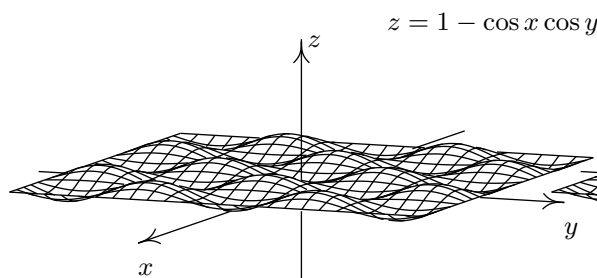


FIGURE 3

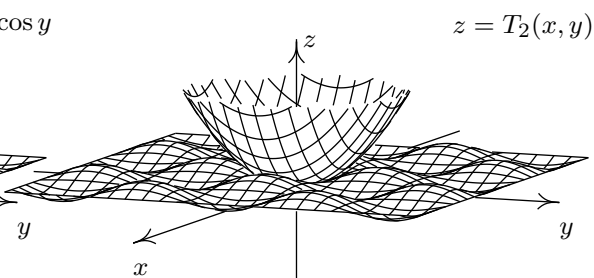
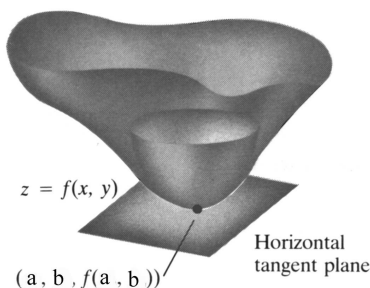


FIGURE 4

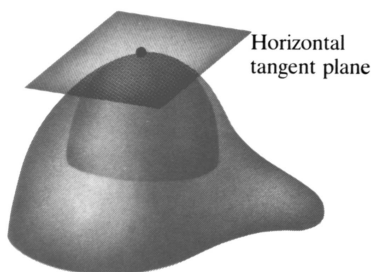
This approach can be applied to any function. Suppose that (a, b) is a critical point of f . Then $f_x(a, b) = 0$ and $f_y(a, b) = 0$ and the Taylor polynomial approximation of f at (a, b) (1) is

$$T_2(x, y) = D + \frac{1}{2}A(x - a)^2 + B(x - a)(y - b) + \frac{1}{2}C(y - b)^2. \tag{2}$$

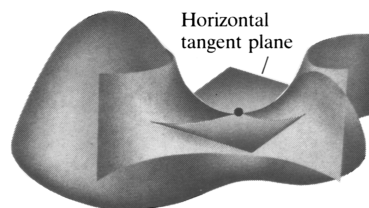
If the graph of T_2 is an elliptic paraboloid that opens upward as in Figure 5, then f has a local minimum at $x = a, y = b$; if the graph of T_2 is an elliptic paraboloid that opens downward as in Figure 6, then f has a local maximum at $x = a, y = b$; and if the graph of T_2 is a hyperbolic paraboloid as in Figure 7, then f has a SADDLE POINT, which is neither a local maximum nor local minimum, at $x = a, y = b$. These geometric ideas are the basis of the Second-Derivative Test with two variables.



Local minimum
FIGURE 5



Local maximum
FIGURE 6



Saddle point
FIGURE 7

Example 2 Figures 8 and 9 show the graph of $f = -x^4 - y^4 - 4xy + \frac{1}{16}$ and its level curves. Use the Second-Derivative Test to classify its critical points.

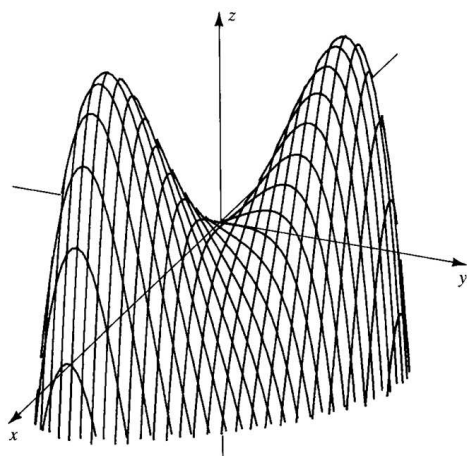


FIGURE 8

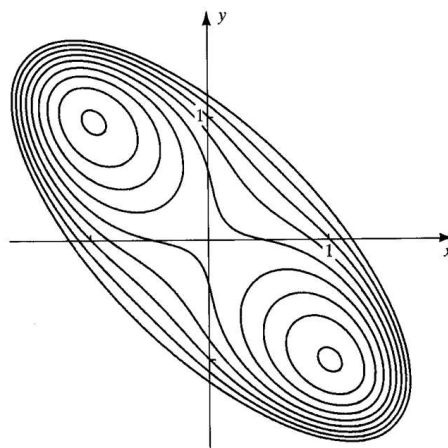


FIGURE 9

Answer: f has a saddle point at $(0, 0)$ and local maxima at $(1, -1)$ and at $(-1, 1)$.

Example 3 Figures 10 and 11 show the graph of $f = -2x^3 - 3y^4 + 6xy^2$ and its level curves. Use the Second-Derivative Test to classify its critical points.

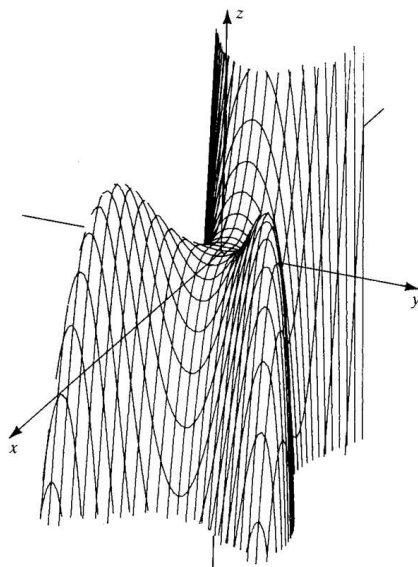


FIGURE 10

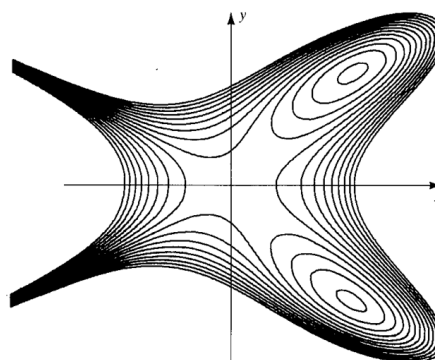


FIGURE 11

Answer: The function has local maxima at $(1, 1)$ and $(1, -1)$. • The Second-Derivative Test fails at $(0, 0)$.

Interactive Examples

Work the following Interactive Examples on Shenk's web page, <http://www.math.ucsd.edu/~ashenk/>.[‡]

Section 15.2: Examples 1–3

[‡]The chapter and section numbers on Shenk's web site refer to his calculus manuscript and not to the chapters and sections of the textbook for the course.