Math 20C. Lecture Examples.

Section 15.4. Integrals in polar and cylindrical coordinates

**Example 1**

(a) Express \( \int \int_R y \, dx \, dy \) as an iterated integral in polar coordinates, where \( R \) is the region bounded by the \( x \)-axis and half cardioid \( r = 1 + \cos \theta, 0 \leq \theta \leq \pi \) in Figure 1.

(b) Evaluate the integral.

![Figure 1](image)

**Answer:** (a) \( \int \int_R y \, dx \, dy = \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=1+\cos \theta} r^2 \sin \theta \, dr \, d\theta \)

(b) \( \int \int_R y \, dx \, dy = \frac{4}{3} \)

**Example 2**

Find the average value of \( f(x, y) = \sqrt{x^2 + y^2 + 1} \) on the circle \( R \): \( x^2 + y^2 \leq 1 \).

**Answer:** Figure A2a • [Average value] = \( \frac{2}{3}(2^{3/2} - 1) \)

![Figure A2](image)

---

†Lecture notes to accompany Section 15.4 of Calculus, Early Transcendentals by Rogawski.
Example 3  Evaluate \( \int \int \int \mathbb{R} xy \, dx \, dy \), where \( \mathbb{R} \) is bounded by the semicircle \( y = \sqrt{x-x^2} \) and the \( x \)-axis.

**Answer:** \( y = \sqrt{x-x^2} \) has the polar equation \( r = \cos \theta \), \( 0 \leq \theta \leq \frac{1}{2} \pi \). Figure A3

\[
\int \int \int \mathbb{R} xy \, dx \, dy = \frac{1}{24} x^2 \quad \text{(This integration procedure is shown in Figure A4.)}
\]

Example 4  (a) Express \( \int \int \int \mathbb{V} 2z \, dx \, dy \, dz \) as an iterated integral in cylindrical coordinates, where \( \mathbb{V} \) is the hemisphere of radius 1 in Figure 2. (b) Evaluate the integral.

**Answer:** (a) \( \int \int \int \mathbb{V} 2z \, dx \, dy \, dz = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \int_{z=0}^{z=\sqrt{1-r^2}} 2rz \, dz \, dr \, d\theta \)

\( \int \int \int \mathbb{V} 2z \, dx \, dy \, dz = \frac{1}{2} \pi \)