Math 20C Supplement

(8/18/08)

Curvature and acceleration in the plane

If a plane curve has the vector equation $\mathbf{R}(t) = \langle x(t), y(t)\rangle$, then at any point P where the velocity vector $\mathbf{v}(t) = \mathbf{R}'(t) = \langle x'(t), y'(t)\rangle$ is not zero, the vector

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}$$

is he vector parallel to the tangent line at P that points in the direction of the curve's orientation. It is the UNIT TANGENT VECTOR to the curve at that point. The unit NORMAL VECTOR $\mathbf{N} = \mathbf{N}(t)$ is the unit vector that is perpendicular to points to the left of $\mathbf{T}(t)$. $\mathbf{N}$ can be found from $\mathbf{T}$ with the following rule.

Rule 1 *The unit normal vector to a curve at a point can be obtained from the unit tangent vector by interchanging its components and then multiplying the first component by -1. Thus, if $\mathbf{T} = \langle a, b\rangle$, then $\mathbf{N} = \langle -b, a\rangle$ (Figure 1).*

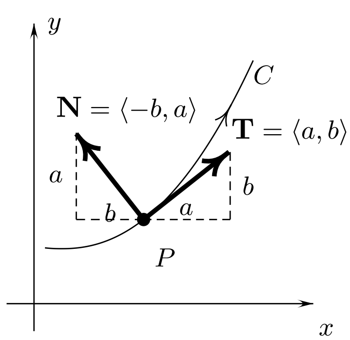


FIGURE 1

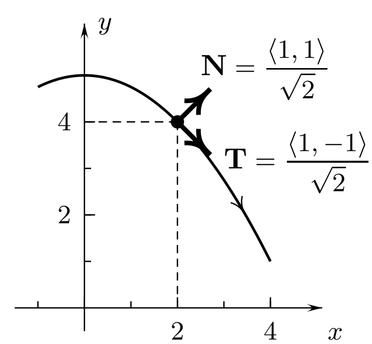


FIGURE 2

Example 1 Find the x- and y-components of $\mathbf{T}$ and $\mathbf{N}$ at $x = 2$ on the parabola $y = 5 - \frac{1}{4}x^2$, oriented from left to right. Then draw the vectors with the curve, using the scales on the axes to measure the components.

SOLUTION Orient the curve from left to right by using x as parameter. The position vector at x is $\mathbf{R}(x) = \langle x, 5 - \frac{1}{4}x^2\rangle$, for which

$$\mathbf{v}(x) = \mathbf{R}'(t) = \frac{d}{dx}\langle x, 5 - \tfrac{1}{4}x^2\rangle = \left\langle \frac{d}{dx}(x), \frac{d}{dx}(5 - \tfrac{1}{4}x^2)\right\rangle = \langle 1, -\tfrac{1}{2}x\rangle.$$

Setting $x = 2$ gives $\mathbf{v}(2) = \langle 1, -1\rangle$. The length of this velocity vector is $|\langle 1, -1\rangle| = \sqrt{1^2 + 1^2} = \sqrt{2}$, so the unit tangent vector is $\mathbf{T} = \langle 1, -1\rangle/\sqrt{2}$. Interchanging the components and then multiplying the first component by -1 gives the unit normal vector $\mathbf{N} = \langle 1, 1\rangle/\sqrt{2}$. The curve and the two vectors are shown in Figure 2. $\square$

Curvature of plane curves

Suppose that a curve has parametric equations,

$$C : x = x(s), y = y(s)$$

with arclength s along the curve as parameter and that $x = x(s)$ and $y = y(s)$ have continuous derivatives in the s-interval being considered. Let $\mathbf{T}(s)$ denote the unit tangent vector at $((x(s), y(s))$ on the curve and let $\phi(s)$ (phi) be the angle of inclination of $\mathbf{T}(s)$ (Figure 3).

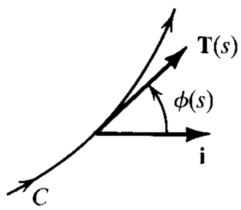


FIGURE 3

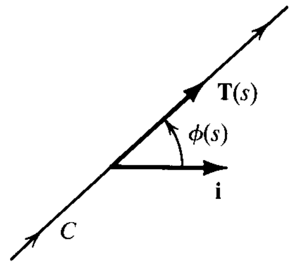


FIGURE 4

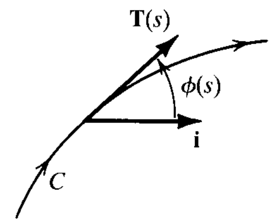


FIGURE 5

Definition 1 *The* CURVATURE κ *(kappa) of* C*:* $x = x(s), y = y(s)$ *at* $((x(s), y(s))$ *is the derivative with respect to arclength* s *of the angle of inclination* $\phi = \phi(s)$ *of the unit tangent vector:*

$$\kappa(s) = \left.\frac{d\phi}{ds}\right|_s. \tag{1}$$

If the curvature **(1)** is positive in an open s-interval, then $\phi = \phi(s)$ is increasing on the interval and that portion of the curve is bending to the left, as in Figure 3. If the curvature is zero in an interval, then $\phi = \phi(s)$ is constant in the interval and that portion of the curve is a straight line, as in Figure 4. If the curvature is negative, then $\phi = \phi(s)$ is decreasing and the curve is bending to the right, as in Figure 5. Moreover, the curve bends sharply where $|\kappa(s)|$ is large and bends gradually where $|\kappa(s)|$ is small.

Curvatures of graphs $y = y(x)$ of functions can be found using the next result.

Theorem 1 *In any interval where* $y = y(x)$ *has a continuous second derivative, the curvature of the graph* C*:* $y = y(x)$*, oriented from left to right, at* $(x, y(x))$ *is*

$$\kappa(x) = \frac{y''(x)}{\{1 + [y'(x)]^2\}^{3/2}}. \tag{2}$$

Proof: Because $y'(x)$ is the slope of the tangent line to $y = y(x)$ at the point with x-coordinate x, the vector $\langle 1, y'(x)\rangle$ has the direction of the unit tangent vector to the curve at that point, so that $\phi = \tan^{-1}[y'(x)]$ is the angle of inclination of the tangent vector. Taking the x-derivative of both sides of this equation gives

$$\left.\frac{d\phi}{dx}\right|_x = \frac{d}{dx}\{\tan^{-1}[y'(x)]\} = \frac{1}{1 + [y'(x)]^2}\,\frac{d}{dx}[y'(x)] = \frac{y''(x)}{1 + [y'(x)]^2}. \tag{3}$$

The length of the graph from $x = a$ to $x > a$ with a fixed number a is

$$s(x) = \int_a^x \sqrt{1 + [y'(t)]^2}\, dt.$$

Part 2 of the Fundamental Theorem of Calculus (derivatives of integrals) gives

$$\left.\frac{ds}{dx}\right|_x = \frac{d}{dx}\int_a^x \sqrt{1+[y'(t)]^2}\,dt = \sqrt{1+[y'(x)]^2}.$$

Because this derivative is positive, $s = s(x)$ has an inverse $x = x(s)$ whose derivative dx/ds is the reciprocal of ds/dx:

$$\left.\frac{dx}{ds}\right|_s = \frac{1}{\sqrt{1+[y'(x)]^2}}. \tag{4}$$

.

Equations **(3)** and **(4)** with the Chain Rule yield

$$\kappa(s) = \left.\frac{d\phi}{ds}\right|_s = \left.\frac{d\phi}{dx}\right|_x \left.\frac{dx}{ds}\right|_s = \frac{y''(x)}{\{1+[y'(x)]^2\}^{3/2}}.\ \textbf{QED}$$

Example 2 What is the curvature at the origin of the parabola $y = -x^2$, oriented from left to right?

SOLUTION For $y(x) = -x^2$, we have $y'(x) = -2x$ and $y''(x) = -2$. Since $y'(0) = 0$, and $y''(0) = -2$, the curvature **(3)** in this case is $\kappa(0) = \dfrac{-2}{(1+0^2)^{3/2}} = -2$. □

The derivation of Theorem 1 can be modified to give the following result for curves given by parametric equations:

Theorem 2 *The curvature at $(x(t), y(t))$ on the curve C: $x = x(t), y = y(t)$ is*

$$\kappa(t) = \frac{x'(t)y''(t) - y'(t)x''(t)}{\{[x'(t)]^2 + [y'(t)]^2\}^{3/2}}. \tag{5}$$

Example 3 What is the curvature at $t = \frac{1}{2}\pi$ of the ellipse $x = 3\cos t, y = 5\sin t$?

SOLUTION For $x(t) = 3\cos t$ and $y(t) = 5\sin t$, we have $x'(t) = -3\sin t, x''(t) = -3\cos t$, $y'(t) = 5\cos t$ and $y''(t) = -5\sin t$. At $t = \frac{1}{2}\pi$, these derivatives have the values $x' = -3, x'' = 0, y' = 0$, and $y'' = -5$, so the curvature **(5)** is

$$\kappa(\tfrac{1}{2}\pi) = \frac{(-3)(-5) - 0(0)}{(3^2+0^2)^{3/2}} = \tfrac{15}{27} = \tfrac{5}{9}.\ \square$$

Example 4 What is the curvature **(a)** of the circle $x = \rho\cos t, y = \rho\sin t$ of radius $\rho > 0$ oriented counterclockwise and **(b)** of the circle $x = \rho\cos(-t), y = \rho\sin(-t)$ of radius ρ oriented clockwise?

SOLUTION **(a)** Since ρ is constant, $x' = \dfrac{d}{dt}(\rho\cos t) = -\rho\sin t$ and $x'' = \dfrac{d}{dt}(-\rho\sin t) = -\rho\cos t$, while $y' = \dfrac{d}{dt}(\rho\sin t) = \rho\cos t$ and $y'' = \dfrac{d}{dt}(\rho\cos t) = -\rho\sin t$. Then because $\sin^2 t + \cos^2 t = 1$, we obtain from **(5)**

$$\kappa = \frac{x'y'' - y'x''}{[(x')^2 + (y')^2]^{3/2}} = \frac{\rho^2(\cos^2 t + \sin^2 t)}{[\rho^2(\sin^2 t + \cos^2 t)]^{3/2}} = \frac{1}{\rho}.$$

(b) Similarly, for $x = \rho\cos(-t), y = \rho\sin(-t)$, we have $x' = \dfrac{d}{dt}[\rho\cos(-t)] = \rho\sin(-t)$ and $x'' = \dfrac{d}{dt}[\rho\sin(-t)] = -\rho\cos(-t)$ while $y' = \dfrac{d}{dt}[\rho\sin(-t)] = -\rho\cos(-t)$ and $y'' = \dfrac{d}{dt}[\rho\cos(-t)] = \rho\sin(-t)$. Consequently,

$$\kappa = \frac{x'y'' - y'x''}{[(x')^2 + (y')^2]^{3/2}} = \frac{-\rho^2[\sin^2(-t) + \cos^2(-t)]}{[\rho^2(\cos^2(-t) + \sin^2(-t))]^{3/2}} = -\frac{1}{\rho}. \square$$

Circles of curvature

Recall that the tangent line at a point P on a curve is the line that best approximates the curve near that point and that the tangent line is the limiting position of the secant line through P and a nearby point Q as Q approaches P. If the curvature of the curve is not zero at P, then we can obtain a better approximation of the curve near P by using its CIRCLE OF CURVATURE at that point, which is defined in the next theorem.

Theorem 3 **(a)** *Suppose that the curvature κ of a plane curve C at the point P is defined and is not zero. Consider the circle through P and nearby points Q, and R on the curve (Figure 6). As the points Q and R approach P, the circle approaches a circle of radius $\rho = 1/|\kappa|$, which is tangent to the curve at P. If κ is positive, the curve is bending to the left and the circle is to the left of the curve relative to its orientation (Figure 7). If κ is negative, the curve is bending to the right and the circle is to the right of the curve relative to its orientation (Figure 8).*

(b) *If the curvature κ at P is zero, then the circle through P, Q, and R approaches the tangent line at P as Q and R approach P.*

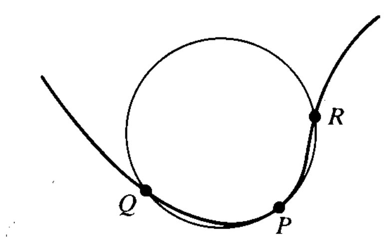
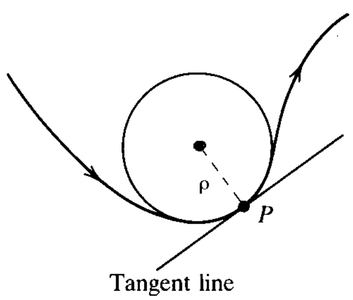
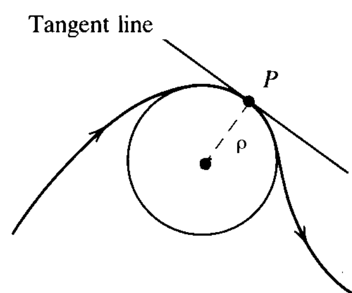


FIGURE 6



The circle of curvature
$\kappa = 1/\rho > 0$

FIGURE 7



The circle of curvature
$\kappa = -1/\rho < 0$

FIGURE 8

The circle in part (a) of Theorem 3 is the CIRCLE OF CURVATURE of the curve at P. Its center is the CENTER OF CURVATURE of the curve at P, and its radius ρ is the RADIUS OF CURVATURE of the curve at P. If $\kappa = 0$, as in part (b) of the theorem, then the tangent line is considered to be the circle of curvature and the radius of curvature is said to be infinite.

The proof of Theorem 3 requires fairly complicated calculations using l'Hopital's Rule and is omitted.

Theorem 3 is illustrated by Example 4, where we found that the curvature of a circle of radius ρ equals $1/\rho$ if the circle is oriented counterclockwise and equals $-1/\rho$ if the curve is oriented clockwise. The theorem holds in this case because the circle of curvature of a circle at any point on it is the circle itself.

Example 5 The centers of curvature at the points P and Q on the curve in Figure 9 are indicated by the nearby dots. What is the approximate curvature of the curve **(a)** at P and **(b)** at Q?

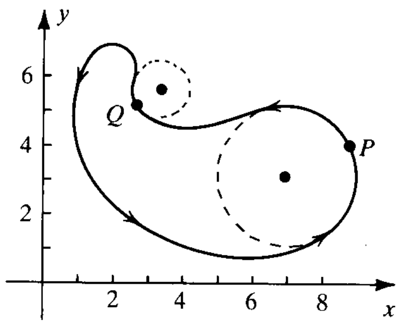


FIGURE 9

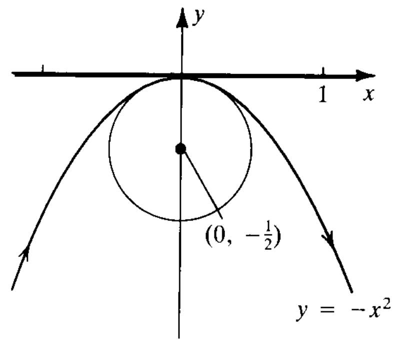


FIGURE 10

SOLUTION **(a)** The radius of curvature ρ at P is the distance to the dot at its left and, measured with the scales on the coordinate axes, is approximately 2. Also, the curve is bending toward the left near P, so that $\kappa = 1/\rho \approx \frac{1}{2}$ at P.
(b) The radius of curvature ρ at Q is approximately 1 and the curve is bending toward the right near Q. Consequently, $\kappa = -1/\rho \approx -1$ at Q. □

Example 6 Draw the parabola $y = -x^2$ of Example 2 and its circle of curvature at the origin.

SOLUTION In Example 2 we found that the curvature of the parabola at the origin is -2, so the radius of curvature there is $1/|-2| = \frac{1}{2}$. Because the curve $y = x^2$ is tangent to the x-axis at the origin, the circle of curvature is also tangent to the x-axis at the origin. The circle of curvature is below the x-axis because κ is negative and the curve is bending to the right. The center of the circle is at the point $(0, -\frac{1}{2})$, as shown in Figure 10. □

Tangential and normal components of acceleration

If no force is applied to an object, then it moves in a straight line at a constant speed. In order for the object to speed up or slow down or for its path to bend, force must be applied to it.

To analyze how forces can affect motion, we suppose that an object of mass m is moving in an xy-plane and the total force on it at time t is $\mathbf{F} = \mathbf{F}(t)$. It the object's velocity vector at time t is $\mathbf{v} = \mathbf{v}(t)$, then its ACCELERATION VECTOR $\mathbf{a} = \mathbf{a}(t)$ is the derivative

$$\mathbf{a} = \mathbf{v}' \tag{6}$$

and by one of NEWTON'S LAWS OF MOTION

$$\mathbf{F} = m\mathbf{a}. \tag{7}$$

.

Because of equation **(7)**, we can relate a object's path to the force applied to it by relating its path to its acceleration vector. We use the following result.

Theorem 4 *Suppose that an object moving in an xy-plane is at $x = x(t), y = y(t)$ at time t for $a \le t \le b$ and that $x = x(t)$ and $y = y(t)$ have continuous second derivatives. Let $s = s(t)$ be the length of the object's path from time a to time t and let $\mathbf{T} = \mathbf{T}(t)$ and $\mathbf{N} = \mathbf{N}(t)$ be the unit tangent and normal vectors to its path at time t Then the object's acceleration vector at time t can be expressed in the form*

$$\mathbf{a} = \frac{d^2s}{dt^2}\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N}. \tag{8a}$$

or in prime notation with the prime denoting t-derivatives (Figure 11),

$$\mathbf{a} = s''\mathbf{T} + \kappa\left(s'\right)^2\mathbf{N}. \tag{8b}$$

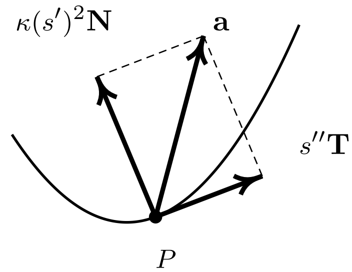


FIGURE 11

The rate of change $\dfrac{ds}{dt}$ of the length of the object's path in **(8)** is the object's (nonnegative) speed. The second derivative $\dfrac{d^2s}{dt^2}$ is the rate of change of the speed with respect to time. We refer to it as the SCALAR ACCELERATION of the object.† It is the rate at which the object is speeding up if it is positive and the negative of the rate at which the object is slowing down if it is negative. Because $\mathbf{T}$ and $\mathbf{N}$ are perpendicular unit vectors, the quantity $\dfrac{d^2s}{dt^2}$ in **(8)** is the TANGENTIAL COMPONENT of the object's acceleration vector and $\kappa\left(\dfrac{ds}{dt}\right)^2$ is the NORMAL COMPONENT.

Proof of Theorem 4: Since $s = s(t)$ is the distance that the object has traveled up to time t, $\dfrac{ds}{dt}$ is its speed. The object's velocity vector is the product of its speed and the unit tangent vector to its path:

$$\mathbf{v} = \frac{ds}{dt}\,\mathbf{T}. \tag{9}$$

We differentiate this equation with respect to t and obtain, with a Product Rule for vectors,

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\frac{d\mathbf{T}}{dt}. \tag{10}$$

To find the derivative of the tangent vector in **(10)**, we express it in terms of its angle of inclination ϕ as $\mathbf{T} = \langle\cos\phi, \sin\phi\rangle$. Then the corresponding unit normal vector is $\mathbf{N} = \langle -\sin\phi, \cos\phi\rangle$, so that

$$\begin{aligned}\frac{d\mathbf{T}}{d\phi} &= \frac{d}{d\phi}\langle\cos\phi, \sin\phi\rangle \\ &= \langle -\sin\phi, \cos\psi\rangle = \mathbf{N}.\end{aligned} \tag{11}$$

†The "scalar acceleration" is generally called "acceleration" in everyday English.

.

By the Chain Rule, the rate of change of $\mathbf{T}$ with respect to t equals its rate of change with respect to ϕ, multiplied by the rate of change of ϕ with respect to s, multiplied by the rate of change of s with respect to t:

$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{d\phi}\,\frac{d\phi}{ds}\,\frac{ds}{dt}.$$

Since $\dfrac{d\mathbf{T}}{d\phi} = \mathbf{N}$ by **(11)** and $\dfrac{d\phi}{ds} = \kappa$ by Definition 1, the last equation gives

$$\frac{d\mathbf{T}}{dt} = \kappa\frac{ds}{dt}\mathbf{N}.$$

With this formula, **(15)** gives

$$\mathbf{a} = \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\left(\kappa\frac{ds}{dt}\mathbf{N}\right) = \frac{d^2s}{dt^2}\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N}$$

as stated in the theorem. **QED**

You can remember formula **(8)** from the special cases of motion on a line and motion at constant speed on a circle. If the motion is on line, then κ is identically zero and equation **(8)** becomes

$$\mathbf{a} = \frac{d^2s}{dt^2}\mathbf{T}.$$

In this case, the acceleration vector is the scalar acceleration multiplied by the unit tangent vector.

If the motion is at constant speed on a circle, then $\dfrac{d^2s}{dt^2}$ is identically zero, and **(8)** reads

$$\mathbf{a} = \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N}.$$

In this case, the acceleration vector is directed toward the center of the circle and its magnitude is equal to the absolute value of the curvature multiplied by the square of the speed.

In the general case, equation **(8)** states that the acceleration vector is equal to that of an object moving on the tangent line to the path with scalar acceleration $\dfrac{d^2s}{dt^2}$, plus that of an object moving around the circle of curvature with speed $\dfrac{ds}{dt}$.

Example 7 Figure 11 shows an insect's path and the tangent and normal lines to its path at a point P where the radius of curvature is four centimeters. When the insect is at P, it is moving six centimeters per second and is speeding up five centimeters per second^2. Draw its approximate acceleration vector at that point, using the scales on the axes to measure the components.

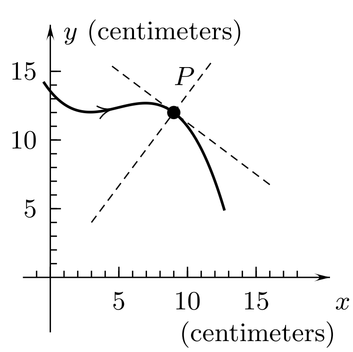


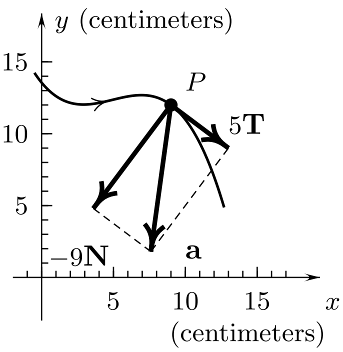
FIGURE 11

SOLUTION We are given that $s' = 6$ centimeters per second and $s'' = 5$ centimeters per second^2 at P. Also, since the curve is bending to the right at P and the radius of curvature is 4, the curvature at P is $\kappa = -\frac{1}{4}$. Equation **(8)** gives

$$\begin{aligned}\mathbf{a} &= s''\mathbf{T} + \kappa(s')^2\mathbf{N} = 5\mathbf{T} - \tfrac{1}{4}(6^2)\mathbf{N}\\ &= 5\mathbf{T} - 9\mathbf{N}.\end{aligned}$$

Since the curve in Figure 11 is oriented toward the right, $\mathbf{T}$ and $5\mathbf{T}$ are tangent to the path at P and point down to the right. The normal vector $\mathbf{N}$ is the unit vector perpendicular to $\mathbf{T}$ that points to its left, so it points up to the right, and $-9\mathbf{N}$ points down to the left. Draw the rectangle with one corner at P and with sides sides formed by $5\mathbf{T}$ and $-9\mathbf{N}$, as in Figure 12. The vector $\mathbf{a}$ lies along its diagonal. □

FIGURE 12



Example 8 When an object is at a point P in an xy-plane, its velocity vector is $\mathbf{v} = \langle 2, 3\rangle$ feet per second, it is slowing down 2 feet per second^2, and the curvature of its path is $\frac{1}{13}$ feet^{-1}. What is its acceleration vector at that moment?

SOLUTION The object's speed at P is the magnitude of its velocity vector, $s' = |\mathbf{v}| = |\langle 2, 3\rangle|$ $= \sqrt{2^2 + 3^2} = \sqrt{13}$ feet per second. Since the object is slowing down 2 feet per second^2, its scalar acceleration is $s'' = -2$ feet per second^2. We are given that the curvature of the object's path is $\kappa = \frac{1}{13}$.

The unit tangent vector to the path is the unit vector in the direction of the object's velocity vector:

$$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\langle 2, 3\rangle}{\sqrt{13}}$$

Since the unit normal vector points is perpendicular to and points to the left of $\mathbf{T}$, it is $\mathbf{N} = \dfrac{\langle -3, 2\rangle}{\sqrt{13}}$.

Putting all this information in equation **(8)** yields

$$\begin{aligned}\mathbf{a} &= s''\mathbf{T} + \kappa(s')^2\mathbf{N} = -2\left(\frac{\langle 2, 3\rangle}{\sqrt{13}}\right) + \tfrac{1}{13}(\sqrt{13})^2\left(\frac{\langle -3, 2\rangle}{\sqrt{13}}\right)\\ &= \frac{-2\langle 2, 3\rangle + \langle -3, 2\rangle}{\sqrt{13}} = \frac{\langle -7, -4\rangle}{\sqrt{13}} \text{ feet per second}^2.\square\end{aligned}$$

Example 9 An object is traveling counterclockwise around the circle $x^2 + y^2 = 9$ of radius 3 feet in Figure 13. Figure 13 shows the acceleration vector of the object $\mathbf{a}$ (feet per second^2) when it is at the point $P = (0, 3)$. (The components of the vector are measured with the scales on the axes.) How fast is the object traveling and at what rate is it speeding up or slowing down at that point?

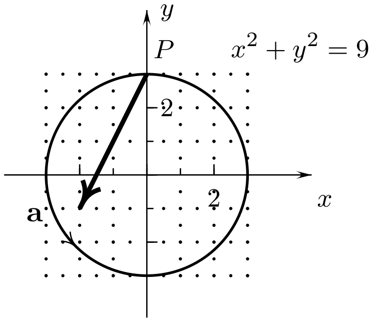


FIGURE 13

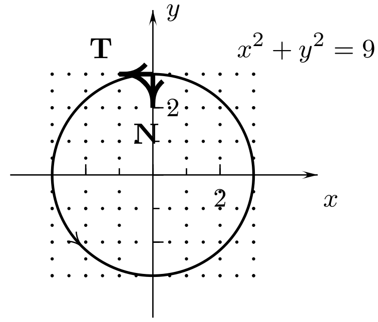


FIGURE 14

SOLUTION Figure 13 shows that $\mathbf{a} = \langle -2, -4 \rangle$. The unit tangent and normal vectors are $\mathbf{T} = \langle -1, 1 \rangle$ and $\mathbf{N} = \langle 0, -1 \rangle$, as shown in Figure 14. Therefore,

$$\begin{aligned} \mathbf{a} &= \langle -2, -4 \rangle \\ &= 2\langle -1, 0 \rangle + 4\langle 0, -1 \rangle \\ &= 2\mathbf{T} + 4\mathbf{N}. \end{aligned} \tag{12}$$

Since the circle has radius 3 and is oriented counterclockwise, its curvature is $\kappa = \frac{1}{3}$ foot^{-1} and equation **(8)** gives

$$\mathbf{a} = s''\mathbf{T} + \tfrac{1}{3}(s')^2\mathbf{N}. \tag{13}$$

Comparing equations **(12)** and **(13)** shows that $s'' = 2$ and $\frac{1}{3}(s')^2 = 4$. The second of these equations implies that $(s')^2 = 12$, so that $s = \sqrt{12} = 2\sqrt{3}$. The object is traveling $2\sqrt{3}$ feet per second and is speeding up 2 feet per second^2. □

Exercises

1. What is the curvature at $x = 1$ of the curve $y = x^3$, oriented from left to right?

Answer: $\kappa = \dfrac{6}{10^{3/2}}$

2. When a ball is at the point P in an xy-plane, its velocity vector is $\mathbf{v} = \langle 8, 6 \rangle$ meters per second, it is speeding up 5 meters per second^2, its path is bending toward the right of its tangent vector, and the radius of curvature of its path is 10 meters. What is its acceleration vector at that moment?

Answer: $s' = |\mathbf{v}| = 10$ meters per second • $s'' = 5$ meters per second^2 • $\kappa = -\frac{1}{10}$ • $\mathbf{T} = \left\langle \frac{4}{5}, \frac{3}{5} \right\rangle$ • $\mathbf{N} = \left\langle -\frac{3}{5}, \frac{4}{5} \right\rangle$ • $\mathbf{a} = \langle 10, -5 \rangle$ meters per second^2

3. Figure 15 shows an object's path and the tangent and normal lines to the path at a point P where the radius of curvature is 8 feet. When the object is at P, it is moving $\sqrt{40}$ feet per second and is speeding up eight feet per second^2. Draw its approximate acceleration vector at that point, using the scales on the axes to measure the components.

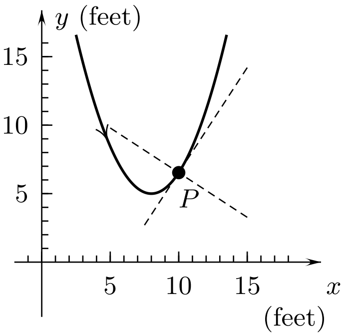


FIGURE 15

Answer: $\mathbf{a} = 8\mathbf{T} + 5\mathbf{N}$ • Figure A3

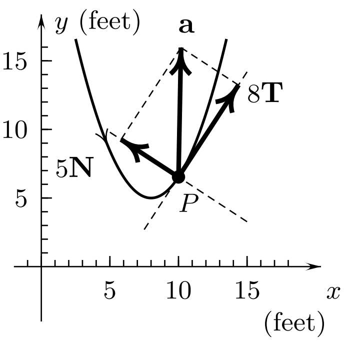


Figure A3

4. What is the acceleration vector of an object at a point where its vector velocity is $\langle 2, -1\rangle$ meters per second if it is speeding up six meters per second^2 and the curvature of its path is $-\frac{1}{5}$ meters^{-1} at that point?

Answer: [Speed] $= \sqrt{5}$ meters per second • $\mathbf{T} = \dfrac{\langle 2,-1\rangle}{\sqrt{5}}$ • $\mathbf{N} = \dfrac{\langle 1,2\rangle}{\sqrt{5}}$ • $\mathbf{a} = \dfrac{\langle 11,-8\rangle}{\sqrt{5}}$ meters per second^2

5. When an object is at a point P, its acceleration vector is $\mathbf{a} = -50\mathbf{T} + 300\mathbf{N}$ feet per second^2, where $\mathbf{T}$ and $\mathbf{N}$ are the unit tangent and normal vectors to its path at P. What is the curvature of its path at P and at what rate is it speeding up or slowing down if its speed is 10 feet per second?

Answer: The curvature is 3 feet^{-1} and the object is slowing down 50 feet per second^2.

6. An object is traveling at a constant speed counterclockwise around the circle $x^2 + y^2 = 4$ in an xy-plane with distances measured in feet. **(a)** What is the curvature of its path? **(b)** What is its speed if its acceleration vector satisfies $\mathbf{a} = 8\mathbf{N}$ at each point on its path?

Answer: (a) $\kappa = \frac{1}{2}$ foot^{-1} **(b)** [Speed] = 4 feet per second

Interactive Examples

Work the following Interactive Examples on Shenk's web page, http//www.math.ucsd.edu/~ashenk/:[†]

Section 13.4: Examples 1–5, 7, 8

[†]The chapter and section numbers on Shenk's web site refer to his calculus manuscript and not to the chapters and sections of the textbook for the course.