Math 20C (Shenk). Summer, 2011. Exam 2 Solution.

1. **(a)** $\mathbf{N} = \dfrac{\langle 4,3\rangle}{|\langle 4,3\rangle|} = \langle \frac{4}{5}, \frac{3}{5}\rangle$ • $\mathbf{T} = \langle \frac{3}{5}, -\frac{4}{5}\rangle$ since $\mathbf{T}$ points to the right of $\mathbf{N}$ • $s' = 10$ •
$\mathbf{v} = s'\mathbf{T} = 10\langle \frac{3}{5}, -\frac{4}{5}\rangle = \langle 6, -8\rangle$ feet per second
(b) $s'' = -5$ • $\rho = 5$ • $\kappa = -\frac{1}{5}$ since the curve bends to the right •
$\mathbf{a} = s''\mathbf{T} + \kappa(s')^2\mathbf{N} = -5\langle \frac{3}{5}, -\frac{4}{5}\rangle + (-\frac{1}{5})(10)^2\langle \frac{4}{5}, \frac{3}{5}\rangle = \langle -3, 4\rangle - \langle 16, 12\rangle = \langle -19 - 8\rangle$ feet per second^2

2. The surface is the graph of $f(x,y) = e^{-x}\sin y$ • One explanation: The vertical cross sections perpendicular to the x-axis where x is constant are sine curves $z = c\sin y$ with constants c, and the vertical cross sections perpendicular to the y-axis where y is constant are exponential curves $z = ce^{-x}$.

3. **(a)** $A(10,60)$ is the equivalent human age of a dog that is 10 years old and weighs 60 pounds.
(b) $A_w(10,60)$ is the rate of change with respect to weight of the equivalent human age of a dog that is ten years old and weighs sixty pounds. •
$A_w(10,60) \approx \dfrac{A(10,80) - A(10,50)}{80-50} = \dfrac{66-60}{30} = \frac{1}{5}$ years per pound

4. $g(2,1) = 2(1) = 2$ • The level curve is $xy = 2$ or equivalently, $y = 2/x$. •
$\nabla g = \langle g_x, g_y\rangle = \langle y, x\rangle$ • $\nabla g(2,1) = \langle 1, 2\rangle$ • Figure A4

5. $\dfrac{\partial}{\partial y}[e^{2x+3y}] = e^{2x+3y}\dfrac{\partial}{\partial y}(2x+3y) = 3e^{2x+3y}$ •
$\dfrac{\partial^2}{\partial x\partial y}e^{2x+3y}] = \dfrac{\partial}{\partial x}[3e^{2x+3y}] = 3e^{2x+3y}\dfrac{\partial}{\partial x}(3x+3y) = 6e^{2x+3y}$

6. $f(x,y) = \sin x + \cos y$ • $f_x = \cos x$ • $f_y = -\sin y$ • $f_x(1,1) = \cos(1)$ • $f_y(1,1) = -\sin(1)$ • For $P = (1,1)$ and $O = (0,0)$, $\overrightarrow{PO} = \langle 0-1, 0-1\rangle = \langle -1,-1\rangle$ • $\mathbf{u} = \dfrac{\overrightarrow{PO}}{|\overrightarrow{PO}|} = \dfrac{\langle -1,-1\rangle}{\sqrt{2}} = \langle u_1, u_2\rangle$ •
$D_{\mathbf{u}}f(1,1) = f_x(1,1)u_1 + f_y(1,1)u_2 = \dfrac{-\cos(1) + \sin(1)}{\sqrt{2}}$

7. Set $g(x,y,z) = x^2 + y^3 + z^4$ • $\nabla g = \langle 2x, 3y^2, 4z^3\rangle$ • $\nabla g(1,1,1) = \langle 2,3,4\rangle$ is perpendicular to the surface at $(1,1,1)$. • $|\langle 2,3,4\rangle| = \sqrt{2^2+3^2+4^2} = \sqrt{29}$ • $\mathbf{u} = \pm\dfrac{\langle 2,3,4\rangle}{|\langle 2,3,4\rangle|} = \pm\dfrac{\langle 2,3,4\rangle}{\sqrt{29}}$

8. Denote the width of the front and back of the box by x, the length of the sides by y, and the height by z, all measured in feet, as in Figure 2 • The area of the bottom of the box is xy, the combined area of the front and back is $2xz$, and the combined area of the sides is $2yz$. •
[Total area] $= xy + 2yz + 2xz$ square feet • Because the volume xyz of the box is to be $\frac{1}{2}$ cubic foot,
$xyz = \frac{1}{2}$ and $z = \dfrac{1}{2xy}$ • [Area] $= A(x,y) = xy + 2x\left(\dfrac{1}{2xy}\right) + 2y\left(\dfrac{1}{2xy}\right) = xy + \dfrac{1}{y} + \dfrac{1}{x}$ •

$$\begin{cases} A_x = \dfrac{\partial}{\partial x}[xy + y^{-1} + x^{-1}] = y - x^{-2} = y - \dfrac{1}{x^2} \\ A_y = \dfrac{\partial}{\partial y}[xy + y^{-1} + x^{-1}] = x - y^{-2} = x + \dfrac{1}{y^2} \end{cases} \bullet \text{ Critical points: } \begin{cases} y - \dfrac{1}{x^2} = 0 \\ x - \dfrac{1}{y^2} = 0 \end{cases} \bullet$$

The first equation gives $y = 1/x^2$, which, when substituted into the second equation, yields
$x = \dfrac{1}{(1/x^2)^2}$ or $x = x^4$ • $x(1-x^3) = 0$ • Since x cannot be zero, $x = 1$ • $y = \dfrac{1}{x^2} = 1$ • $z = \dfrac{1}{2xy} = \frac{1}{2}$

9. $f = x^4 - 4x + xy - y$ • $f_x = 4x^3 - 4 + y, f_y = x - 1$ • Solve $\begin{cases} 4x^3 - 4 + y = 0 \\ x - 1 = 0. \end{cases}$ •
The second equation implies that $x = 1$ and then the first implies that $y = 0$. • Critical point: $(1,0)$ • $f_{xx} = 12x^2, f_{xy} = 1, f_{yy} = 0$ • $A = f_{xx}(1,0) = 12, B = f_{xy}(1,0) = 1, C = f_{yy}(1,0) = 0$ • $AC - B^2 = 12(0) - 1^2 = -1$ is negative. • The critical point is a saddle point.

10. $f = x^2 + 2y^2$ • $g = 2x + y$ • $\nabla f = \lambda\nabla g$ • $< 2x, 4y >= \lambda < 2, 1 >$ • $2x = 2\lambda, 4y = \lambda y$ •
$x = \lambda, y = \frac{1}{4}\lambda$ • $2\lambda + \frac{1}{4}\lambda = 9$ • $\frac{9}{4}\lambda = 9$ • $\lambda = 4$ • $x = \lambda = 4, y = \frac{1}{4}\lambda = 1$ •
The minimum is $f(4,1) = 4^2 + 2(1)^2 = 18$