Math 20C (Shenk). Quiz 2 Solution. August 22, 2011.

1. $y = e^{2x}$ • $y' = 2e^{2x}$ • $y'' = 4e^{2x}$ • $y'(0) = 2$ • $y''(0) = 4$ •

$$\kappa(0) = \frac{y''(0)}{\{1 + [y'(0)]^2\}^{3/2}} = \frac{4}{(1+2^2)^{3/2}} = \frac{4}{5^{3/2}}$$

2. $\mathbf{a} = \frac{d}{dt}(\mathbf{v}) = \frac{d}{dt}\left(\frac{ds}{dt}\mathbf{T}\right) = \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\frac{d\mathbf{T}}{dt}$ by the definition of the velocity vector and the Product Rule.

• $\frac{d\mathbf{T}}{d\phi} = \frac{d}{d\phi}\langle\cos\phi, \sin\phi\rangle = \langle -\sin\phi . \cos\phi\rangle = \mathbf{N}$ • $\frac{d\phi}{ds} = \kappa$ •

By the Chain Rule, $\frac{d\mathbf{T}}{dt} = \frac{dT}{d\phi}\frac{d\phi}{ds}\frac{ds}{dt} = \kappa\frac{ds}{dt}\mathbf{N}$ •

$$\mathbf{a} = \frac{d^2s}{dt^2}\mathbf{T} + \left(\frac{ds}{dt}\right)\left(\kappa\frac{ds}{dt}\mathbf{N}\right) = \frac{d^2s}{dt^2}\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N}$$

3. **(a)** The radius of curvature ρ is the distance $\overline{OC}$ from the origin to the the center of curvature $C = (4,3)$. • $\rho = \sqrt{4^2+3^2} = \sqrt{25} = 5$ • Since the curve bends to the left of the tangent vector, κ is positive. • $\kappa = \frac{1}{\rho} = \frac{1}{5}$

(b) Since the tangent vector points down toward the right, the normal vector points toward the center of curvature $C = (4,3)$ and is the unit vector with the direction of $\overrightarrow{OC} = \langle 4,3\rangle$ • $\mathbf{N} = \frac{\langle 4,3\rangle}{|\langle 4,3\rangle|} = \langle \frac{4}{5}, \frac{3}{5}\rangle$

(c) $\mathbf{T} = \langle \frac{3}{5}, -\frac{4}{5}\rangle$ • $\mathbf{a} = s''\mathbf{T} + \kappa(s')^2\mathbf{N} = 10\langle\frac{3}{5}, -\frac{4}{5}\rangle + \frac{1}{5}(5)^2\langle\frac{4}{5}, \frac{3}{5}\rangle = \langle 6,-8\rangle + \langle 4,3\rangle = \langle 10,-5\rangle$ meters per minute^2 • Figure A3

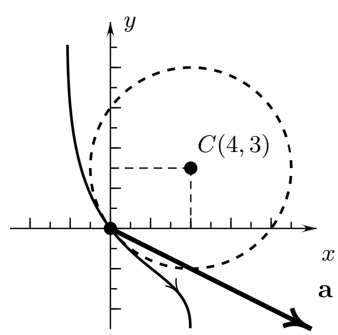

Figure A3

4. **(a)** From the level curve $L = 3$ to the level curve $L = 2$ along $y = 10$:

$\Delta L = 2 - 3 = -1$ and $\Delta x \approx 5$ • $L_x(20,10) \approx \frac{\Delta L}{\Delta x} \approx \frac{-1}{5} = -0.2$

(b) From the level curve $L = 2$ to the level curve $L = 3$ on the line through $(20,10)$ and the origin, $\Delta L = 3 - 2 = 1$ and $\Delta s \approx 4$. • [Directional derivative] $\approx \frac{\Delta L}{\Delta s} \approx \frac{1}{4} = 0.25$

5. The graph of $G(x, y) = -1 - x^2 - y^2$ is a circular paraboloid which opens downward and has its vertex at -1 on the z-axis. • Figure A5

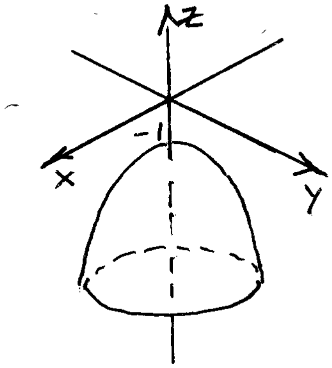

Figure A5

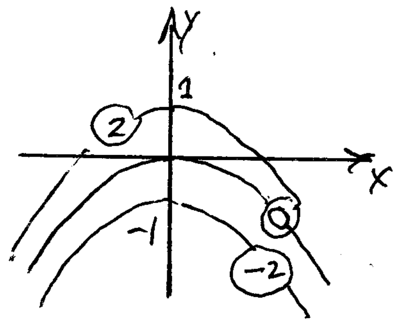

Figure A6

6. $2y + x^2 = c \iff y = \frac{1}{2}x^2 + \frac{1}{2}c$ • One answer, using $c = 0, -2$, and 2: Figure A6

7. $M_x = \dfrac{M(6,1) - M(1,1)}{6-1} = \dfrac{30-20}{5} = 4$ • $M_y = \dfrac{M(1,6) - M(1,1)}{6-1} = \dfrac{0-10}{5} = -2$ • $M(1,1) = 10$ • $M(x, y) = 10 + 4(x-1) - 2(y-1)$

8. $\dfrac{\partial}{\partial y}\sin(x^2+y^2) = \cos(x^2+y^2)\dfrac{\partial}{\partial y}(x^2+y^2) = 2y\cos(x^2+y^2)$ •

$\dfrac{\partial^2}{\partial x \partial y}\sin(x^2+y^2) = \dfrac{\partial}{\partial x}[2y\cos(x^2+y^2)] = 2y[-\sin(x^2+y^2)\dfrac{\partial}{\partial x}(x^2+y^2)] = -4xy\sin(x^2+y^2)$

9. $f(x,y) = x^2y^3$ • $f_x = 2xy^3$ • $f_y = 3x^2y^2$ • $f(5,-1) = 5^2(-1)^3 = -15$ • $f_x(5,-1) = 2(5)(-1)^3 = -10$ • $f_y(5,-1) = 3(5)^2(-1)^2 = 75$ • Tangent plane: $z = -15 - 19(x-5) + 75(y+1)$

10. $f = xy$ • $f_x = y$ • $f_y = x$ • $f_x(3,-1) = -1$ • $f_y(3,-1) = 3$ • For $P = (3,-1)$ and $Q = (7,2)$, $\overrightarrow{PQ} = \langle 7-3, 2-(-1)\rangle = \langle 4,3\rangle$ • $\mathbf{u} = \dfrac{\langle 4,3\rangle}{|\langle 4,3\rangle|} = \langle \frac{4}{5}, \frac{3}{5}\rangle$ •

$D_{\mathbf{u}}f(3,-1) = u_1f_x + u_2f_y = (\frac{4}{5})(-1) + (\frac{3}{5})(3) = 1$

11. $g = \frac{1}{3}x^3 + 3\ln y + e^z$ • $\nabla g = \langle g_x, g_y, g_z\rangle = \langle x^2, \dfrac{3}{y}, e^z\rangle$ • $\nabla g(1,3,0) = \langle 1,1,1\rangle$ •

[Maximum directional derivative of g at $(1,3,0)$] $= |\nabla g(1,3,0)| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$