## Math 20C. Lecture Examples.

## Section 12.1. Vectors in the plane ${ }^{\dagger}$

Definition 1 A VECTOR $\mathbf{v}$ represents a nonnegative number and, if the number is not zero, a direction. The number associated with the vector $\mathbf{v}$ is called its LENGTH or MAGNITUDE and is denoted $|\mathbf{v}|$. The vector of zero length is the ZERO VECTOR and is denoted $\mathbf{O}$. It has no direction.

Vectors are generally denoted by bold-faced letters, like the $\mathbf{v}$ in Definition 1, or by letters with arrows over them, as in the symbol $\overrightarrow{P Q}$.

Vectors are represented in drawings by arrows, where in each case the length of the arrow is the magnitude of the vector, measured with a scale that might or might not be the scale used on the coordinate axes. The direction of the arrow is the direction associated with the vector (Figure 1). The same vector can be represented by different arrows in different locations (Figure 2), provided that the different arrows are parallel, have the same lengths, and point in the same direction.


FIGURE 1


FIGURE 2

If a vector is represented by an arrow in an $x y$-plane as in Figure 3, then the $x$ - and $y$-COMPONENTS of the vector are the changes in the $x$ - and $y$-coordinates from the base to the tip of the arrow, measured with the scale used for measuring the length of the arrow. If the $x$-component of $\mathbf{v}$ is $a$ and the $y$-component is $b$, as shown in Figure 6 with positive $a$ and $b$, we write $\mathbf{v}=\langle a, b\rangle$. The length of a vector can be calculated from its $x$ - and $y$-components by using the Pythagorean Theorem:

$$
|\mathbf{v}|=|\langle a, b\rangle|=\sqrt{a^{2}+b^{2}}
$$

FIGURE 3

$a$

[^0]A nonzero vector $\mathbf{v}$ in an $x y$-plane can also be described by giving its length $|\mathbf{v}|$ and its angle of inclination, which is an angle $\theta$ from the positive $x$-direction to the vector (Figure 4 ). Then the components of the vector are given by

$$
\mathbf{v}=|\mathbf{v}|\langle\cos \theta, \sin \theta\rangle=\langle | \mathbf{v}|\cos \theta,|\mathbf{v}| \sin \theta\rangle
$$

This formula is a consequence of the definition of the sine and cosine functions. The zero vector has zero components: $\mathbf{O}=\langle 0,0\rangle$.

FIGURE 4


Example 1 Find the $x$ - and $y$-components of the vector $\mathbf{u}$ of length 10 with angle of inclination $\frac{5}{6} \pi$.
Answer: Figure A1a • $\mathbf{u}=\langle-5 \sqrt{3}, 5\rangle \bullet$ Figure A1b


Figure A1a


Figure A1b

Example $2 \quad$ Find an angle of inclination of the vector $\mathbf{w}=\langle 3,4\rangle$.
Answer: Figure A2. - $\theta=\tan ^{-1}\left(\frac{4}{3}\right)$

Figure A2


## Adding vectors and multiplying vectors by numbers

Definition 2 For any vectors $\mathbf{v}=\langle a, b\rangle$ and $\mathbf{w}=\langle c, d\rangle$ and any number $\lambda$,

$$
\begin{gather*}
\mathbf{v}+\mathbf{w}=\langle a, b\rangle+\langle c, d\rangle=\langle a+c, b+d\rangle  \tag{1}\\
\lambda \mathbf{v}=\lambda\langle a, b\rangle=\langle\lambda a, \lambda b\rangle \tag{2}
\end{gather*}
$$

Equation (1) has two geometric interpretations that are illustrated in Figures 5 and 6 for vectors with positive components: If we place the base of $\mathbf{w}$ at the tip of $\mathbf{v}$, then the vector $\mathbf{v}+\mathbf{w}$ is given by the arrow from the base of $\mathbf{v}$ to the tip of $\mathbf{w}$, as shown in Figure 8. If we place the bases of $\mathbf{v}$ and $\mathbf{w}$ together, as in Figure 9 and complete the parallelogram with those vectors as sides, then $\mathbf{v}+\mathbf{w}$ is represented by the arrow with base at the bases of $\mathbf{v}$ and $\mathbf{w}$ that goes along the diagonal of the parallelogram to the opposite vertex.


FIGURE 5


FIGURE 6

Equation (2) is illustrated in Figure 7. Multiplying the vector $\mathbf{v}$ by a positive number $\lambda$ yields a vector with the same direction as $\mathbf{v}$ whose length is $\lambda$ multiplied by the length of $\mathbf{v}$. Multiplying $\mathbf{v}$ by a negative number $\mu$ yields a vector with the opposite direction as $\mathbf{v}$ whose length is $|\mu|$ multiplied by the length of $\mathbf{v}$.

FIGURE 7


You can think of the difference $\mathbf{w}-\mathbf{v}$ of two vectors as the vector which when added to $\mathbf{v}$ gives $\mathbf{w}$ (Figure 8 ). It the bases of $\mathbf{v}$ and $\mathbf{w}$ are at the same point, then $\mathbf{w}-\mathbf{v}$ is the vector from the tip of $\mathbf{v}$ to the top of $\mathbf{w}$.

FIGURE 8


Example 3 Calculate (a) $\mathbf{v}+\mathbf{w}$ and (b) $\mathbf{w}-\mathbf{v}$ for $\mathbf{v}=\langle 4,1\rangle$ and $\mathbf{w}=\langle 1,3\rangle$. Then draw the four vectors.
Answer: (a) $\mathbf{v}+\mathbf{w}=\langle 5,4\rangle \bullet$ Figure A3a (b) $\mathbf{w}-\mathbf{v}=\langle-3,2\rangle \bullet$ Figure A3b


Figure A3a


Figure A3b

Example $4 \quad$ Write $3\langle 4,-1\rangle-2\langle 10,-5\rangle$ in the form $\langle a, b\rangle$.
Answer: $3\langle 4,-1\rangle-2\langle 10,-5\rangle=\langle-8,7\rangle$

## Displacement and position vectors

The displacement vector from one point $P\left(x_{0}, y_{0}\right)$ to a second point $Q\left(x_{1}, y_{1}\right)$ is denoted $\overrightarrow{P Q}$ and can be represented by an arrow with base at $P$ and tip at $Q$. Its components are obtained by subtracting the coordinates of $P$ from the coordinates of $Q$ :

$$
\overrightarrow{P Q}=\left\langle x_{1}-x_{0}, y_{1}-y_{0}\right\rangle
$$

(See Figure 9.)
Points can be located in an $x y$-plane by using their POSITION VECTORS. The position vector of the point $P(x, y)$ is the displacement vector $\overrightarrow{O P}$

$$
\overrightarrow{O P}=\langle x, y\rangle
$$

from the origin to the point (Figure 10). The components of the position vector are the coordinates of the tip of the vector.


Example $5 \quad$ Three vertices of the parallelogram $P R S Q$ in Figure 11 are $P=(3,4), Q=(7,8)$, and $R=(12,2)$. What are the coordinates of $S ?$

FIGURE 11


Answer: $S=(16,6)$

## Unit vectors

Vectors of length 1 are called Unit VEctors. For any nonzero vector $\mathbf{v}$, the unit vector with the same direction as $\mathbf{v}$ is $\mathbf{e}_{\mathbf{v}}=\frac{\mathbf{v}}{|\mathbf{v}|}$.
Example 6 Give the unit vector $\mathbf{e}_{\mathbf{v}}$ and the vector $\mathbf{w}$ of length 5 with the same direction as $\mathbf{v}=\langle-3,2\rangle$.

$$
\text { Answer: } \mathbf{e}_{\mathbf{v}}=\frac{\langle-3,2\rangle}{\sqrt{13}} \bullet \mathbf{w}=\frac{\langle-15,10\rangle}{\sqrt{13}}
$$

The unit vectors $\langle 1,0\rangle$ and $\langle 0,1\rangle$ in the directions of the positive $x$ - and $y$-axes in Figure 12 are denoted $\mathbf{i}$ and $\mathbf{j}$, respectively. These vectors can be used in place of angular brackets to express a vector in terms of its $x$-and $y$-components by writing (Figure 13)

$$
\mathbf{v}=\langle a, b\rangle=a\langle 1,0\rangle+b\langle 0,1\rangle=a \mathbf{i}+b \mathbf{j} .
$$



FIGURE 12


FIGURE 13

Example $7 \quad$ Express $3(4 \mathbf{i}-\mathbf{j})-2(10 \mathbf{i}-5 \mathbf{j})$ in the form $a \mathbf{i}+b \mathbf{j}$.
Answer: $3(4 \mathbf{i}-\mathbf{j})-2(10 \mathbf{i}-5 \mathbf{j})=-8 \mathbf{i}+7 \mathbf{j}$. (Example 4 is the same calculation with bracket notation for the vectors.)

## Sums of force vectors

It is an empirical fact that two forces $\mathbf{F}$ and $\mathbf{G}$ applied at the same point $P$ on an object have the same effect as their sum $\mathbf{F}+\mathbf{G}$ (Figure 14). Because of this, the sum is called the Resultant of $\mathbf{F}$ and $\mathbf{G}$.

FIGURE 14


Example 8 One man is lifting a boulder with a rod while another is pulling it with a rope as in Figure 3. (a) Find the $x$ - and $y$-components of the two force vectors, with the usual orientation of axes. (b) Find the resultant of the two forces and the approximate decimal values of its magnitude and angle of inclination


FIGURE 15
Answer: (a) [Force exerted by the man with the rod] $=\mathbf{F}=300\left\langle\cos \left(\frac{7}{18} \pi\right), \sin \left(\frac{7}{18} \pi\right)\right\rangle$ pounds.
[Force exerted by the man with the rope] $=\mathbf{G}=150\left\langle\cos \left(\frac{1}{9} \pi\right), \sin \left(\frac{1}{9} \pi\right)\right\rangle$ pounds.
(b) [Resultant] $=\left\langle 300 \cos \left(\frac{7}{18} \pi\right)+150 \cos \left(\frac{1}{9} \pi\right), 300 \sin \left(\frac{7}{18} \pi\right)+150 \sin \left(\frac{1}{9} \pi\right)\right\rangle \doteq\langle 244,333\rangle$ pounds • [Magnitude of the combined force $\doteq 413$ pounds • [Angle of inclination] $\doteq \tan ^{-1}\left(\frac{333}{244}\right) \doteq 0.94$ radians.

## Interactive Examples

Work the following Interactive Examples on Shenk's web page, http//www.math.ucsd.edu/ ashenk/: ${ }^{\dagger}$
Section 12.1: Examples 1, 6, 7

[^1]
[^0]:    ${ }^{\dagger}$ Lecture notes to accompany Section 12.1 of Calculus, Early Transcendentals by Rogawski.

[^1]:    ${ }^{\dagger}$ The chapter and section numbers on Shenk's web site refer to his calculus manuscript and not to the chapters and sections of the textbook for the course.

