Math 20C. Lecture Examples.

Section 12.2. Vectors in three dimensions†

To use rectangular xyz-coordinates in three-dimensional space, we introduce mutually perpendicular x-, y-, and z-axes intersecting at their origins, as in Figure 1. These axes form xy-, xz-, and yz-coordinate planes, which divide the space into eight OCTANTS.

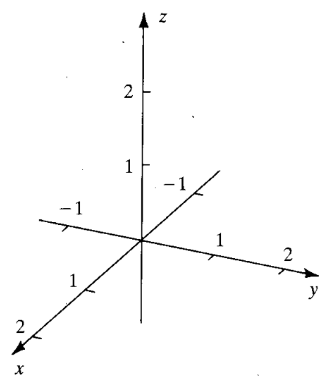


FIGURE 1

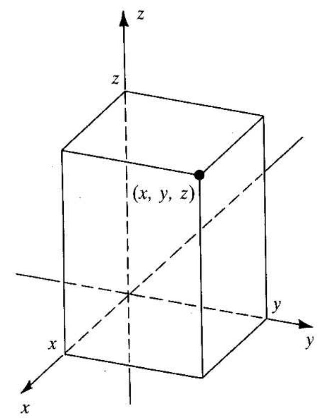


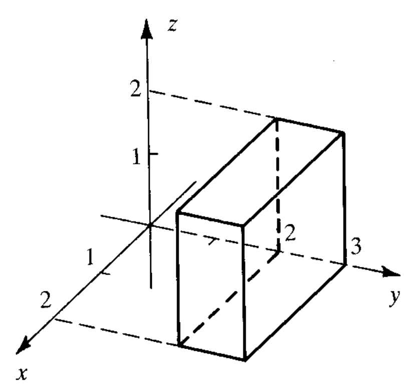
FIGURE 2

The coordinates (x, y, z) of a point in space are determined by planes passing through the point and perpendicular to the coordinate axes (Figure 2).

Example 1 Sketch the box consisting of all points (x, y, z) with $0 \le x \le 2, 2 \le y \le 3$, and $0 \le z \le 2$. What are the coordinates of the eight corners of the box?

Answer: Figure A1. • The corners of its base, ordered counterclockwise, are $(2, 2, 0), (2, 3, 0), (0, 3, 0)$, and $(0, 2, 0)$. • The corners of its top are $(2, 2, 2), (2, 3, 2), (0, 3, 2)$, and $(0, 2, 2)$.

Figure A1



†Lecture notes to accompany Section 12.2 of *Calculus, Early Transcendentals* by Rogawski.

The Pythagorean Theorem and the distance between two points

If a rectangular box has length a, width b, and height c, as in Figure 6, then, by the Pythagorean Theorem for a right triangle, the length of a diagonal of its base is $\sqrt{a^2+b^2}$. Then, because the diagonal of the box is the hypotenuse of a right triangle with base of length $\sqrt{a^2+b^2}$ and height c (Figure 7), its length is the square root of $[\sqrt{a^2+b^2}]^2 + c^2 = a^2+b^2+c^2$. This gives the Pythagorean Theorem in space:

$$\begin{bmatrix}\text{The length of a diagonal of a} \\ \text{rectangular box with sides } a, b, \text{ and } c \text{ is}\end{bmatrix} = \sqrt{a^2+b^2+c^2}.$$

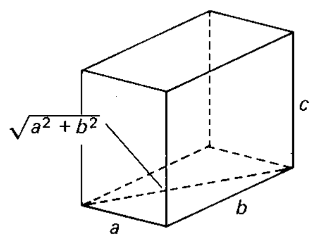


FIGURE 6

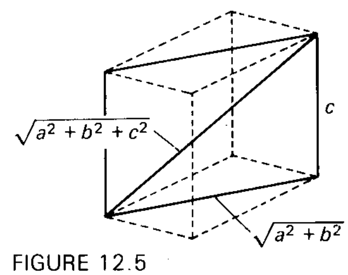


FIGURE 7

Example 2 What is the length of the diagonals of the box from Example 1?

Answer: The length of each of its four diagonals is $\sqrt{1^2+2^2+2^2} = 3$

Because points $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ in xyz-space are at diagonally opposite corners of a rectangular box with sides of lengths $|x_2 - x_1|, |y_2 - y_1|$, and $|z_2 - z_1|$, the distance $\overline{PQ}$ between the points is

$$\overline{PQ} = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}.$$

Example 3 Describe the set of points defined by the equation $(x-1)^2 + (y-2)^2 + (z-3)^2 = 16$.

Answer: $(x-1)^2 + (y-2)^2 + (z-3)^2 = 16$ is the sphere of radius 4 with its center at $(1, 2, 3)$.

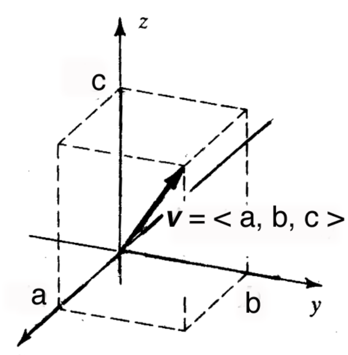
Example 4 Describe the set of points defined by the equation $x^2 + y^2 = 25$.

Answer: $x^2 + y^2 = 25$ is the cylinder of radius 5 with the z-axis as its axis.

Vectors in space

A nonzero vector in xyz- space, like a nonzero vector $\mathbf{v}$ in an xy-plane, represents a positive number and a direction. If we put the base of the vector at the origin, as in Figure 8, then the coordinates (a, b, c) of its tip are the x-, y-, and z-components of the vector and we write $\mathbf{v} = \langle a, b, c\rangle$. The zero vector $\mathbf{0} = \langle 0, 0, 0\rangle$ has zero length and no direction.

FIGURE 8



The Pythagorean Theorem in space shows that the length of the vector $\mathbf{v} = \langle a, b, c\rangle$ is equal to the square root of the sum of the squares of its components:

$$|\mathbf{c}| = |\langle a, b, c\rangle| = \sqrt{a^2 + b^2 + c^2}.$$

The rules for adding two vectors in space and multiplying a vector in space by a real number are analogous to those for vectors in a plane:

Definition 1 *For any vectors* $\mathbf{v} = \langle a_1, b_1, c_1\rangle$ *and* $\mathbf{w} = \langle a_2, b_2, c_2\rangle$ *and any number* λ,

$$\mathbf{v} + \mathbf{w} = \langle a_1, b_1, c_1\rangle + \langle a_2, b_2, c_2\rangle = \langle a_1 + a_2, b_1 + b_2, c_1 + c_2\rangle$$

$$\lambda\mathbf{v} = \lambda\langle a_1, b_1, c_1\rangle = \langle \lambda a_1, \lambda b_1, \lambda c_1\rangle.$$

These operations and the subtraction of vectors in space have the same geometric interpretations as in an xy-plane (see Figures 9 through 12.)

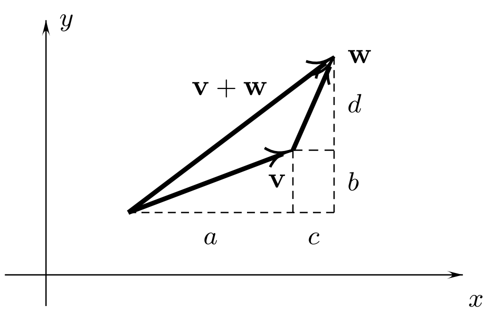


FIGURE 9

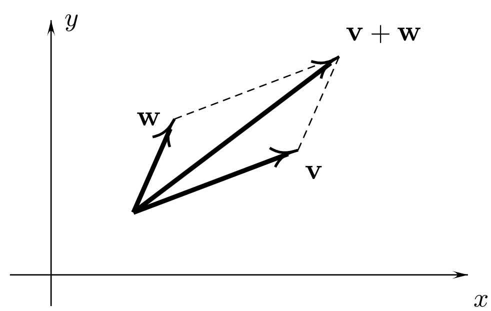


FIGURE 10

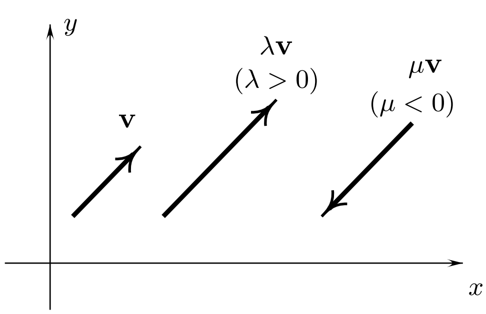


FIGURE 11

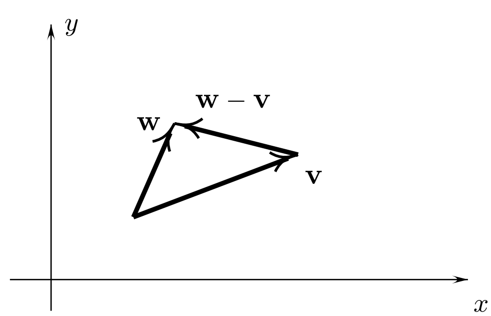


FIGURE 12

The unit vectors $\mathbf{i}, \mathbf{j}$, *and* $\mathbf{k}$

In the last section we expressed the vector $\langle a, b\rangle$ in the plane as $a\mathbf{i} + b\mathbf{j}$ where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the directions of the positive x- and y-axes, respectively. In three dimensions, we also use a third unit vector $\mathbf{k}$ in the direction of the positive z-axis, as in Figure 13. Then $\langle a, b, c\rangle = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ for any a, b, and c (Figure 14).

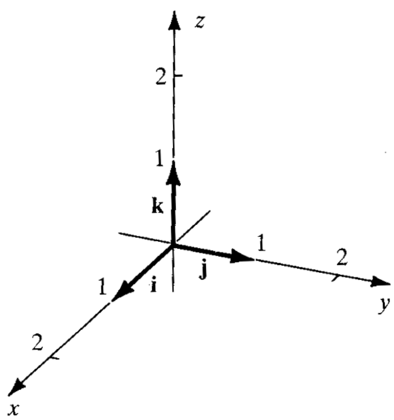


FIGURE 13

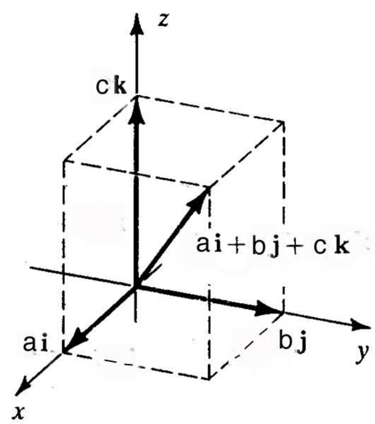


FIGURE 14

Example 5 Write $\mathbf{z} = \mathbf{u} + 2\mathbf{v} + 3\mathbf{w}$ in the form $a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$, where $\mathbf{u} = 3\,\mathbf{i} - \mathbf{j}$, $\mathbf{v} = \mathbf{j} - 3\,\mathbf{k}$ and $\mathbf{w} = \mathbf{i} + \mathbf{k}$.

Answer: $\mathbf{z} = 6\,\mathbf{i} + \mathbf{j} - 3\,\mathbf{k}$

The position vector $\overrightarrow{OP}$ of a point (x, y, z) in space is $\langle x, y, z\rangle$ (Figure 15). The displacement vector $\overrightarrow{PQ}$ from $P = (x_1, y_1, z_1)$ to $Q = (x_2, y_2, z_2)$ is

$$\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1\rangle$$

as is shown in Figure 16 for a case where $y_2 - y_1$ and $z_2 - z_1$ are positive and $x_2 - x_1$ is negative.

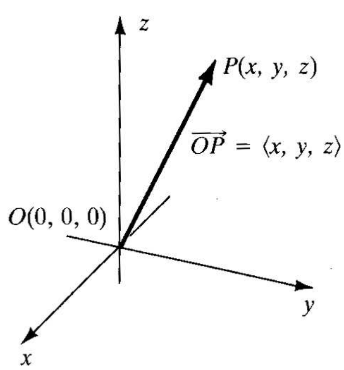
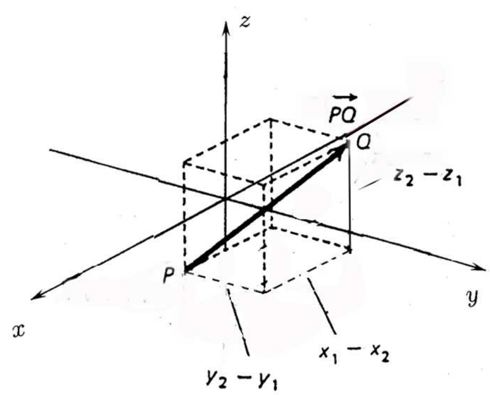


FIGURE 15



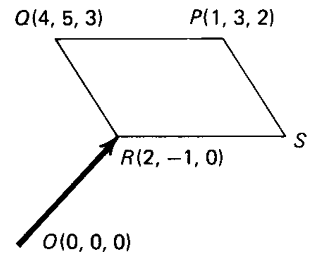
$$\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1\rangle$$

FIGURE 16

Example 6 Three adjacent vertices of a parallelogram $PQRS$ in space are $P = (1, 3, 2)$, $Q = (4, 5, 3)$, and $R = (2, -1, 0)$. What are the coordinates of the point S opposite Q?

Answer: Use the schematic sketch in Figure A6. • $S = (-1, -3, -1)$

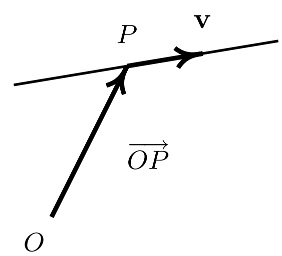
Figure A6



Parametric equations of lines in space

A line in xyz-space can be described by giving the coordinates of a point $P = (x_0, y_0, z_0)$ on it and a nonzero vector $\mathbf{v} = \langle a, b, c\rangle$ parallel to it, as in Figure 17.

FIGURE 17



Theorem 1 *The line L through the point $P = (x_0, y_0, z_0)$ and parallel to the nonzero vector $\mathbf{v} = \langle a, b, c\rangle$ in xyz-space has the parametric equations,*

$$L\colon \begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct. \end{cases}$$

Example 7 Give parametric equations for the line L through the point $(6, 4, 3)$ and parallel to the vector $2\mathbf{i} + 5\mathbf{j} - 7\mathbf{k}$.

Answer: L: $x = 6 + 2t,\ y = 4 + 5t,\ z = 3 - 7t$

Example 8 Give parametric equations for the line L through $P = (5, 3, 1)$ and $Q = (7, -2, 0)$.

Answer: L:$x = 5 + 2t, y = 3 - 5t, z = 1 - t$

Example 9 Find the intersection of the lines L_1: $x = 2 - t, y = 3 + t, z = 4 - 2t$ and L_2: $x = -3 + t, y = -1 + 2t, z = 9 - 3t$

Answer: Intersection: $(0, 5, 0)$

Interactive Examples

Work the following Interactive Examples on Shenk's web page, http//www.math.ucsd.edu/˜ashenk/:[†]

Section 12.3: Examples 1, 2, and 6

Section 12.5: Examples 1 and 2

[†] The chapter and section numbers on Shenk's web site refer to his calculus manuscript and not to the chapters and sections of the textbook for the course.