Math 20C. Lecture Examples.

Section 12.3. The dot product and angles between vectors[†]

Definition 1 *The* DOT PRODUCT *of vectors* $\mathbf{v} = \langle a_1, b_1 \rangle$ *and* $\mathbf{w} = \langle a_2, b_2 \rangle$ *in a coordinate plane is the number*

$$\mathbf{v} \cdot \mathbf{w} = a_1 a_2 + b_1 b_2.$$

If $\mathbf{v} = \langle a_1, b_1, c_1 \rangle$ *and* $\mathbf{w} = \langle a_2, b_2, c_2 \rangle$ *are in* xyz*-space, then*

$$\mathbf{v} \cdot \mathbf{w} = a_1 a_2 + b_1 b_2 + c_1 c_2.$$

Example 1 **Calculate $\mathbf{v} \cdot \mathbf{w}$ for $\mathbf{v} = \langle 6, -2 \rangle$ and $\mathbf{w} = \langle 4, 3 \rangle$.**

Answer: $\mathbf{v} \cdot \mathbf{w} = 18.$

Example 2 **What is $\mathbf{v} \cdot \mathbf{w}$ for $\mathbf{v} = \langle 6, -2, 3 \rangle$ and $\mathbf{w} = \langle 4, 3, -6 \rangle$.**

Answer: $\mathbf{v} \cdot \mathbf{w} = 0$

The dot product satisfies the following rules for any vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ and any number λ:

$$\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$$
$$(\lambda \mathbf{v}) \cdot \mathbf{w}) = \mathbf{v} \cdot (\lambda \mathbf{w}) = \lambda(\mathbf{v} \cdot \mathbf{w})$$
$$(\mathbf{v} + \mathbf{w}) \cdot \mathbf{u} = \mathbf{v} \cdot \mathbf{u} + \mathbf{w} \cdot \mathbf{u}$$
$$\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$$

The dot product is useful because of the next theorem.

Theorem 1 *If neither* $\mathbf{v}$ *nor* $\mathbf{w}$ *is the zero vector, then*

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}||\mathbf{w}| \cos\theta \tag{1}$$

where θ *is an angle between* $\mathbf{v}$ *and* $\mathbf{w}$*.*

An angle between nonzero vectors can be found with the formula,

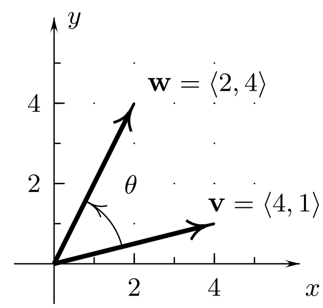
$$\cos\theta = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}||\mathbf{w}|} \tag{2}$$

which comes from **(1)**.

[†]Lecture notes to accompany Section 12.3 of *Calculus, Early Transcendentals* by Rogawski.

Example 3 **Find an angle θ between the vectors $\mathbf{v} = \langle 4, 1\rangle$ and $\mathbf{w} = \langle 2, 4\rangle$ in Figure 1. Give exact and approximate decimal values.**

FIGURE 1



Answer: $\theta = \cos^{-1}\left(\dfrac{12}{\sqrt{17}\sqrt{20}}\right) \doteq 0.862$ radians

Perpendicular vectors

The dot product provides a way to determine whether two vectors are perpendicular.

Theorem 2 *Two vectors* $\mathbf{v}$ *and* $\mathbf{w}$ *are perpendicular if and only if* $\mathbf{v} \cdot \mathbf{w} = 0$.

By convention, the zero vector is considered to be perpendicular to all vectors.

Example 4 **Find the constant k such that the vectors $\langle -3, -1\rangle$ and $\langle k, -2\rangle$ are perpendicular. Then draw the two vectors.**

Answer: $k = \frac{2}{3}$ • The vectors are $\langle \frac{2}{3}, -2\rangle$ and $\langle -3, -1\rangle$. • Figure A4

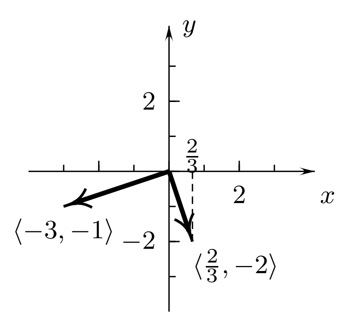


Figure A4

Interactive Examples

Work the following Interactive Examples on Shenk's web page, http//www.math.ucsd.edu/˜ashenk/:[‡]

Section 12.3: Examples 1–3, 6

Section 12.4: Examples 1–6

[‡]The chapter and section numbers on Shenk's web site refer to his calculus manuscript and not to the chapters and sections of the textbook for the course.