Math 20C. Lecture Examples.

Section 12.4. The cross product†

There are two directions perpendicular to two nonzero and nonparallel vectors $\mathbf{v}$ and $\mathbf{w}$ in xyz-space. They are distinguished by using the RIGHT-HAND RULE: A nonzero vector $\mathbf{u}$ perpendicular to $\mathbf{v}$ and $\mathbf{w}$ has the direction given by the right-hand rule from $\mathbf{v}$ toward $\mathbf{w}$ if, when the fingers of a right hand curl from $\mathbf{u}$ toward $\mathbf{v}$, as in Figure 1, the thumb points in the direction of $\mathbf{u}$.

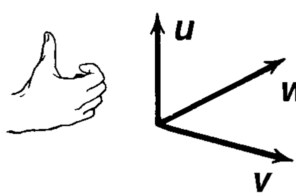

FIGURE 1

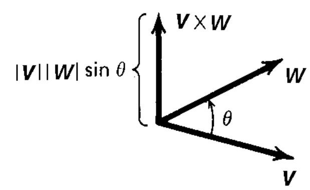

FIGURE 2

The right-hand rule is used in the definition of the cross product of two vectors.

Definition 1 *The cross product* $\mathbf{v} \times \mathbf{w}$ *of nonzero and nonparallel vectors* $\mathbf{v}$ *and* $\mathbf{w}$ *in* xyz*-space is the vector perpendicular to* $\mathbf{v}$ *and* $\mathbf{w}$ *with direction determined by the right-hand rule from* $\mathbf{v}$ *toward* $\mathbf{w}$ *and whose length is*

$$|\mathbf{v} \times \mathbf{w}| = |\mathbf{v}||\mathbf{w}| \sin\theta \tag{1}$$

where θ *is the angle with* $0 < \theta < \pi$ *between* $\mathbf{v}$ *and* $\mathbf{w}$*. (Figure 2). If* $\mathbf{v}$ *or* $\mathbf{w}$ *is the zero vector or they are parallel, then* $\mathbf{v} \times \mathbf{w}$ *is the zero vector.*

The cross product has the properties listed in the next theorem. Notice the minus sign in equation **(2)**.

Theorem 1 *For any vectors* $\mathbf{u}, \mathbf{v}$*, and* $\mathbf{w}$ *in* xyz*-space and any number* λ*,*

$$\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v} \tag{2}$$

$$(\lambda\mathbf{v}) \times \mathbf{w} = \mathbf{v} \times (\lambda\mathbf{w}) = \lambda(\mathbf{v} \times \mathbf{w}) \tag{3}$$

$$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w} \tag{4}$$

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†Lecture notes to accompany Section 12.4 of *Calculus, Early Transcendentals* by Rogawski.

Calculating cross products with determinants

Cross products can be calculated using the notation of DETERMINANTS from linear algebra.

The 2×2 (two-by-two) determinant

$$\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$$

denotes the number $x_1y_2 - x_2y_1$ (Figure 3).

$$\begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} = x_1y_2 - x_2y_1$$

FIGURE 3

Then 3×3 determinants can be calculated with the formula,

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = x_1 \begin{vmatrix} y_2 & y_3 \\ z_2 & z_3 \end{vmatrix} - x_2 \begin{vmatrix} y_1 & y_3 \\ z_1 & z_3 \end{vmatrix} + x_3 \begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix}.$$

Each of the determinants on the right of **(5)** is obtained by crossing out the row and column of one of the numbers in the first row of the 3×3 determinant. The expression on the right equals the first number in the first row of the original determinant, multiplied by the corresponding 2×2 determinant, minus the second number in the first row multiplied by the corresponding 2×2 determinant, plus the third number in the first row multiplied by the corresponding 2×2 determinant. This procedure is caled the EXPANSION of the determinant by its first row (Figure 4).

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = x_1 \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} - x_2 \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} + x_3 \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$$

FIGURE 4

Example 1 Evaluate $\begin{vmatrix} 3 & 2 & 4 \\ -1 & 0 & 6 \\ 5 & 1 & -2 \end{vmatrix}$.

Answer: The given determinant equals 34.

Now we can give a procedure for calculating cross products from the components of the vectors.

Theorem 2 *The cross product of vectors* $\mathbf{v} = \langle a_1, b_1, c_1\rangle$ *and* $\mathbf{w} = \langle a_2, b_2, c_2\rangle$ *is equal to the determinant*

$$\mathbf{v}\times\mathbf{w} = \langle a_1, b_1, c_1\rangle \times \langle a_2, b_2, c_2\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

that is obtained by putting the unit vectors $\mathbf{i}, \mathbf{j}$, *and* $\mathbf{k}$ *in the first row, the components of* $\mathbf{v}$ *in the second row, and the components of* $\mathbf{w}$ *in the third row.*

Example 2 Find the cross product of $\mathbf{v} = \langle 3, 1, -2\rangle$ and $\mathbf{w} = \langle 0, 4, 2\rangle$.

Answer: $\mathbf{v}\times\mathbf{w} = \langle 10, -6, 12\rangle$

Example 3 As a partial check of the result of Example 2, show that each the given vectors is perpendicular to the calculated cross product.

Answer: Let $\mathbf{u} = \langle 10, -6, 12\rangle$ be the calculated cross product. • $\mathbf{v}\cdot\mathbf{u} = 0$ • $\mathbf{w}\cdot\mathbf{u} = 0$

Example 4 Find a nonzero vector perpendicular to $\mathbf{v} = 4\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{w} = 2\mathbf{i} - \mathbf{k}$.

Answer: One answer: The cross product $\mathbf{v}\times\mathbf{w} = \mathbf{i} + 6\mathbf{j} + 2\mathbf{k}$ is perpendicular to $\mathbf{v}$ and $\mathbf{w}$.

Cross products and areas

Theorem 3 **(a)** *If the nonzero vectors* $\mathbf{v}$ *and* $\mathbf{w}$ *with their bases at the same point in* xyz*-space form two sides of a parallelogram, then*

$$[\text{The area of the parallelogram}] = |\mathbf{v}\times\mathbf{w}|.$$

(b) *If the vectors* $\mathbf{v}$ *and* $\mathbf{w}$ *with their bases at the same point in* xyz*-space form two sides of a triangle, then*

$$[\text{The area of the triangle}] = \tfrac{1}{2}|\mathbf{v}\times\mathbf{w}|.$$

Example 5 Find the area of the triangle with vertices $P = (1, 2, 3), Q = (4, 2, 6)$ and $R = (5, 3, 7)$.

Answer: [Area of the triangle] $= \frac{3}{2}\sqrt{2}$

The scalar triple product

The number $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$ is called a SCALAR TRIPLE PRODUCT of the three vectors.[†] It can be calculated as a determinant:

Theorem 3 (The scalar triple product) *For vectors* $\mathbf{u} = \langle a_1, b_1, c_1\rangle$, $\mathbf{v} = \langle a_2, b_2, c_2\rangle$, *and* $\mathbf{w} = \langle a_3, b_3, c_3\rangle$

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

where the rows of the determinant are the components of $\mathbf{u}, \mathbf{v}$, *and* $\mathbf{w}$ *in that order.*

[†] $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})$ is called a scalar triple product because it is a scalar (number) and to distinguish it from the vector triple product $\mathbf{u}\times(\mathbf{v}\times\mathbf{w})$.

Example 6 Calculate $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ for $\mathbf{u} = \langle 3, 3, -1 \rangle, \mathbf{v} = \langle 4, 6, 5 \rangle$, and $\mathbf{w} = \langle 2, 2, -1 \rangle$.

Answer: $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = -2$

Scalar triple products and volumes

If we position three vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ with their bases at the same point, then they form adjacent edges of a PARALLELEPIPED, as in Figure 6, or adjacent edges of a TETRAHEDRON, as in Figure 7. The volumes of these solids can be calculated with the scalar triple product.

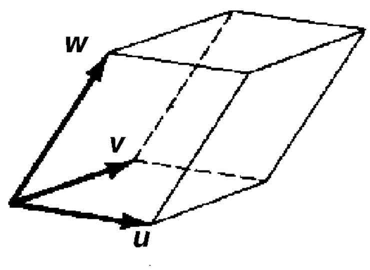

FIGURE 6

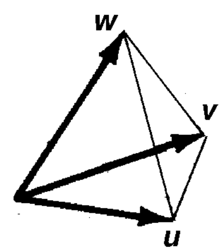

FIGURE 7

Theorem 5 **(a)** *If three adjacent edges of a parallelepiped are formed by the the vectors* $\mathbf{u}, \mathbf{v}$, *and* $\mathbf{w}$, *then*

$$[\text{The volume of the parallelepiped}] = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|.$$

(b) *If the vectors* $\mathbf{u}, \mathbf{v}$, *and* $\mathbf{w}$ *form adjacent sides of a tetrahedron, then*

$$[\text{The volume of the tetrahedron}] = \tfrac{1}{6}|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|.$$

Example 7 What is the volume of the parallelepiped with vertex $P = (1, 1, 1)$ and adjacent vertices $Q = (4, 4, 0), R = (5, 7, 6)$, and $S = (3, 3, 0)$?

Answer: [Volume of the parallelepiped] $= 2$

Interactive Examples

Work the following Interactive Examples on Shenk's web page, http//www.math.ucsd.edu/˜ashenk/:†

Section 12.4: Examples 1–7

†The chapter and section numbers on Shenk's web site refer to his calculus manuscript and not to the chapters and sections of the textbook for the course.