Math 20C. Lecture Examples.

Sections 11.2, 13.2, and 13.3. Calculus of vector-valued functions†

Example 1 Find $\lim_{t\to 1}\langle t^2-3,\ e^{3t},\ \ln t\rangle$.

Answer: $\lim_{t\to 1}\langle t^2-3,\ e^{3t},\ \ln t\rangle = \langle -2,\ e^3,\ 0\rangle$

Example 2 What is $\lim_{t\to 3}\mathbf{r}(t)$ if $\mathbf{r}(t)=\langle -t, t^2-5\rangle$?

Answer: $\lim_{t\to 3}\mathbf{r}(t)=\langle -3, 4\rangle$

Example 3 Find the derivative $\frac{d}{dt}\langle t^2-3,\ e^{3t},\ \ln t\rangle$.

Answer: $\frac{d}{dt}\langle t^2-3,\ e^{3t},\ \ln t\rangle = \left\langle 2t,\ 3e^{3t},\ \frac{1}{t}\right\rangle$

Example 4 What is the derivative $\mathbf{r}'(\frac{1}{3}\pi)$ for $\mathbf{r}(t)=2\cos t\,\mathbf{i}+4\sin t\,\mathbf{j}$?

Answer: $\mathbf{r}'(\frac{1}{3}\pi)=-\sqrt{3}\mathbf{i}+2\mathbf{j}$

Velocity vectors and speed

Suppose that an object moving in an xy-plane has position vector $\mathbf{r}=\mathbf{r}(t)$ at time t (Figure 1). For small positive Δt, the displacement vector $\mathbf{r}(t+\Delta t)-\mathbf{r}(t)$ points from one point to the other on the curve in the direction of the object's motion. Dividing $\mathbf{r}(t+\Delta t)-\mathbf{r}(t)$ by the positive number Δt to form the difference quotient,

$$\frac{\mathbf{r}(t+\Delta t)-\mathbf{r}(t)}{\Delta t} \tag{1}$$

changes the length of the vector but not its direction (Figure 2). This vector also lies along a secant line and points in the direction of the object's motion for $\Delta t<0$ because then $\mathbf{r}(t+\Delta t)-\mathbf{r}(t)$ points in the opposite direction and dividing it by a negative number reverses its direction.

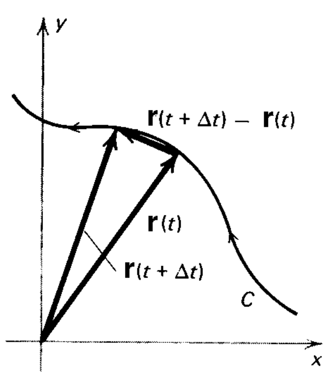


FIGURE 1

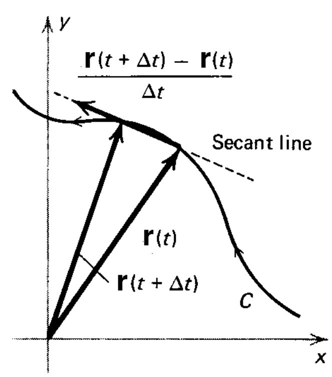


FIGURE 2

Suppose that the derivative $\mathbf{r}'(t)$ exists. Then $\mathbf{r}=\mathbf{r}(t)$ is continuous at t and the tip of $\mathbf{r}(t+\Delta t)$ approaches the tip of $\mathbf{r}(t)$ as $\Delta t\to 0$. Moreover, if the derivative is not the zero vector, then the secant line approaches a line, through the tip of $\mathbf{r}(t)$ and parallel to $\mathbf{r}'(t)$. We define the line to be the tangent line to the curve and refer to $\mathbf{r}'(t)$ as the VELOCITY VECTOR $\mathbf{v}(t)$ of the moving object at that point (Figure 3). We also define the length of the velocity vector to be the object's SPEED (Figure 4) because for positive Δt, the length

$$\left|\frac{\mathbf{r}(t+\Delta t)-\mathbf{r}(t)}{\Delta t}\right| = \frac{|\mathbf{r}(t+\Delta t)-\mathbf{r}(t)|}{\Delta t}$$

†Lecture notes to accompany Sections 11.2, 13.2, and 13.3 of *Calculus, Early Transcendentals* by Rogawski.

of the difference quotient **(1)** is approximately equal to the distance the object travels from time t to time $t + \Delta t$, divided by the time taken.

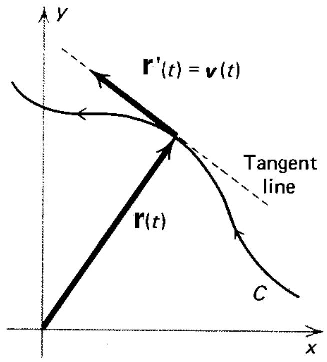


FIGURE 3

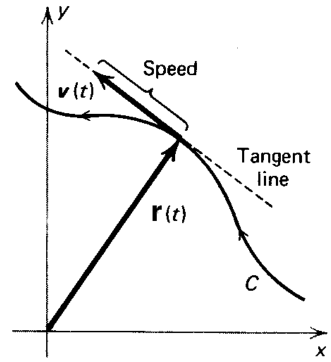


FIGURE 4

The same reasoning can be applied to a curve in space. We are led to the next definition.

Definition 1 **(a)** *If an object that is moving in an xy-plane or in xyz-space has the position vector $\mathbf{r} = \mathbf{r}(t)$ at time t, then its* VELOCITY VECTOR *at time t is the derivative,*

$$\mathbf{v}(t) = \mathbf{r}'(t)$$

and the length of this vector is the object's SPEED *at time t:*

$$[\text{Speed}] = |\mathbf{v}(t)| = \left|\mathbf{r}'(t)\right|.$$

(b) *If the velocity vector is not $\mathbf{0}$, then the tangent line to the curve C: $\mathbf{r} = \mathbf{r}(t)$ at the point with position vector $\mathbf{r}(t)$ is the line through the point and parallel to the velocity vector.*

The terms "velocity vector" and "speed," as defined in Definition 1b are also used when t is a parameter that is not time. Definition 1b gives the same tangent lines as other definitions we have used in any cases where the other definitions apply. The velocity vector of an object is drawn with its base at the object's position at the time being considered.

Example 5 Find the velocity vector to the ellipse C: $x = 5\cos t, y = 3\sin t$ at $t = \frac{1}{4}\pi$. Then draw the ellipse and the velocity vector.

Answer: $\mathbf{v}(\frac{1}{4}\pi) = \langle -\frac{5}{2}\sqrt{2}, \frac{3}{2}\sqrt{2}\rangle$ • Figures A3a and A3b

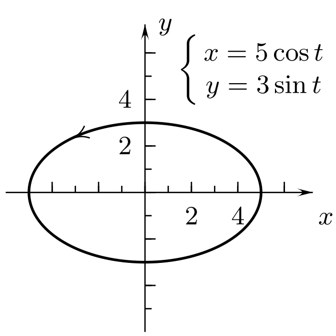
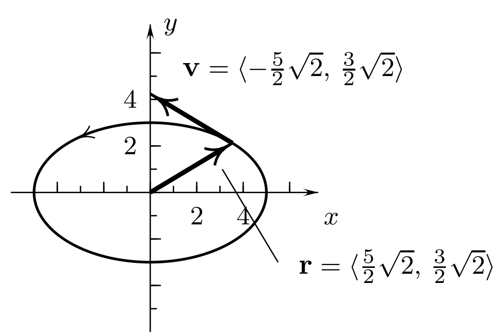


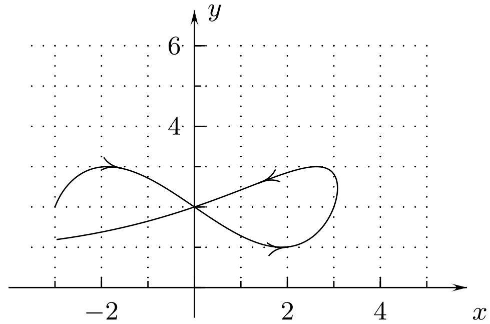
Figure A3a



Figue A3b

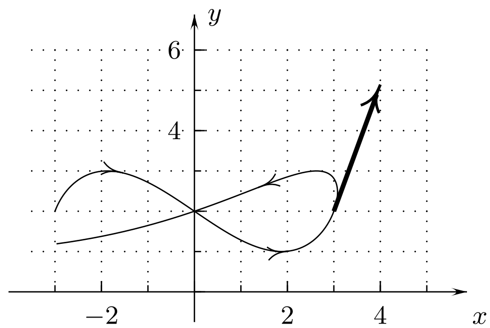
Example 6 The curve C: $x = 4t - t^3, y = 2 - \sin(\pi t), -1 \le t \le 2.3$ is in Figure 5. **(a)** Find its velocity vector at $t = 1$ and draw it with the curve using the scales on the axes to measure the components. **(b)** What is the speed at $t = 1$?

FIGURE 5



Answer: (a) [Velocity at $t = 1$] $= \langle 1, \pi\rangle \doteq \langle 1, 3.14\rangle$ • Put the base of the velocity vector at $(x(1), y(1)) = (3, 2)$. • Figure A6 **(b)** [Speed at $t = 1$] $= \sqrt{1 + \pi^2}$

Figure A6



Example 7 The graphs of differentiable functions $x = x(t)$ and $y = y(t)$ are given in Figures 6 and 7, and the curve C: $\mathbf{R}(t) = \langle x(t), y(t)\rangle, 0 \le t \le 6$ is in Figure 8. Draw on a copy of C the approximate velocity vectors $\mathbf{R}'(t)$ at $t = 2$ and $t = 4$.

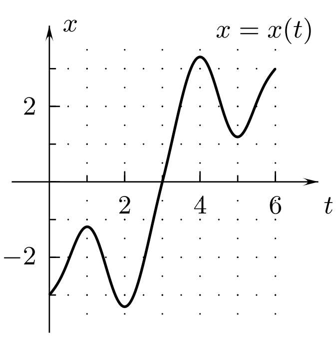


FIGURE 6

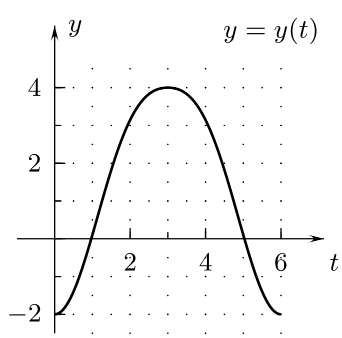


FIGURE 7

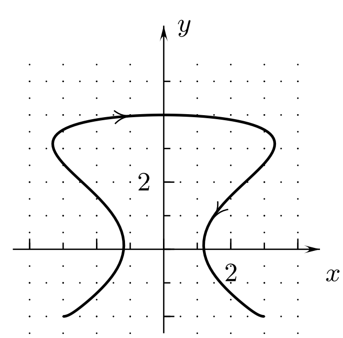


FIGURE 8

Answer: Figures A7a and A7b. • $\mathbf{v}(2) \approx \langle 0, 2\rangle$, $\mathbf{v}(4) \approx \langle 0, -2\rangle$ • $\mathbf{R}(2) \approx \langle -3.3, 3.1\rangle$ • $\mathbf{R}(4) \approx \langle 3.3, 3.1\rangle$ • Figure A7c

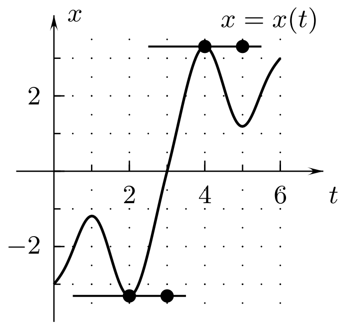


Figure A7a

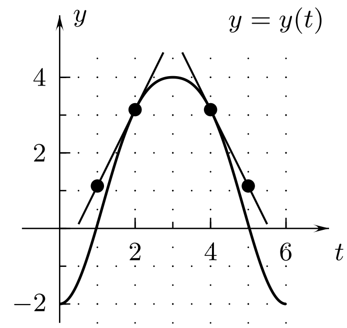


Figure A7b

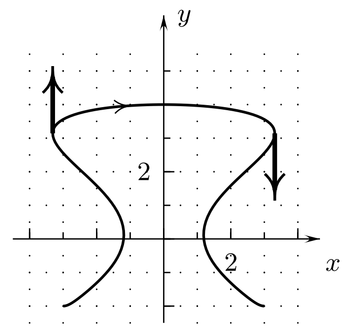
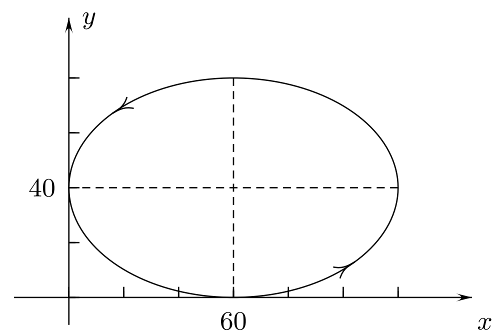


Figure A7c

Example 8 A robot moving in an xy-plane with distances measured in meters is at $(120, 40)$ at $t = 0$ (minutes) and its velocity vector is $\mathbf{v}(t) = \langle -120\sin(2t),\ 80\cos(2t)\rangle$ (meters per minute) at time t . Find the robot's position vector $\mathbf{R} = \mathbf{R}(t)$ and describe its path.

Answer: $\mathbf{R}(t) = \langle 60\cos(2t) + 60,\ 40\sin(2t) + 40\rangle$ (meters) • The path is the ellipse in Figure A10 with center at $(60, 40)$, horizontal axis of length 120, and vertical axis of length 80.

Figure A8



Lengths of curves

The formula,

$$[\text{Length}] = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

for the length of the graph $y = f(x), a \le x \le b$ of a function $y = f(x)$ with a piecewise-continuous derivative in the interval $[a, b]$, has the following generalization for curves given by parametric equations.

Definition 2 *Suppose that a curve in the xy-plane or in xyz-space is given by* C: $\mathbf{r} = \mathbf{r}(t)$, $a \le t \le b$, *where the derivative* $\mathbf{v}(t) = \mathbf{r}'(t)$ *is piecewise continuous in* $[a, b]$. *Then the length of the curve is given by the integral,*

$$[\text{Length}] = \int_a^b |\mathbf{v}(t)|\ dt = \int_a^b \left|\mathbf{r}'(t)\right|\ dt.$$

Notice that the length of the curve equals the integral of the speed $|\mathbf{v}(t)|$ of an object that traverses it. In the case of a curve $C : \mathbf{r}(t) = \langle x(t), y(t)\rangle$ in an xy-plane,

$$[\text{Length}] = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$

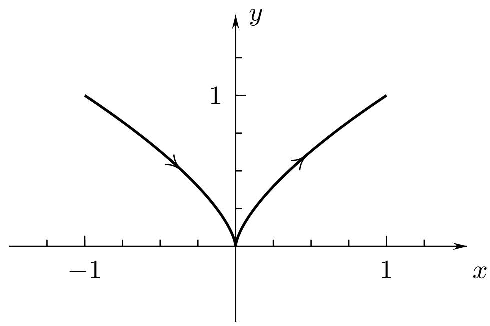
and for a curve $C : \mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ in space

$$[\text{Length}] = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt.$$

Example 9 Find the length of the curve $x = t^3, y = t^2, -1 \le t \le 1$ in Figure 9.

$$\begin{cases} x = t^3 \\ y = t^2 \\ -1 \le t \le 1 \end{cases}$$

FIGURE 9

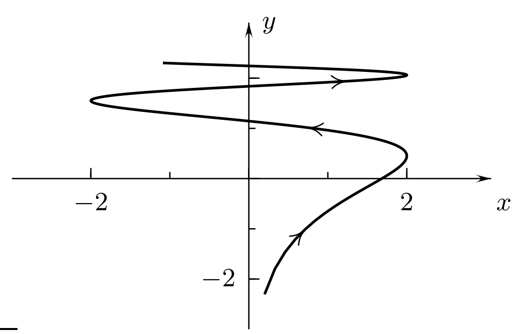


Answer: $[\text{Length}] = \frac{2}{27}(13^{3/2} - 8)$

Example 10 Give a definite integral that equals the length of C: $x = 2\sin t$, $y = \ln t$, $0.1 \le t \le 10$ in Figure 10.

$$\begin{cases} x = 2\sin t \\ y = \ln t \\ 0.1 \le t \le 10 \end{cases}$$

FIGURE 10



Answer: $[\text{Length}] = \displaystyle\int_{0.1}^{10} \sqrt{4\cos^2 t + t^{-2}}\, dt$

Interactive Examples

Work the following Interactive Examples on Shenk's web page, http//www.math.ucsd.edu/˜ashenk/:[†]

Section 13.2: Examples 1–5

[†]The chapter and section numbers on Shenk's web site refer to his calculus manuscript and not to the chapters and sections of the textbook for the course.