(8/30/08)

Math 20C. Lecture Examples.

Section 14.1, Part 1. Functions of two variables†

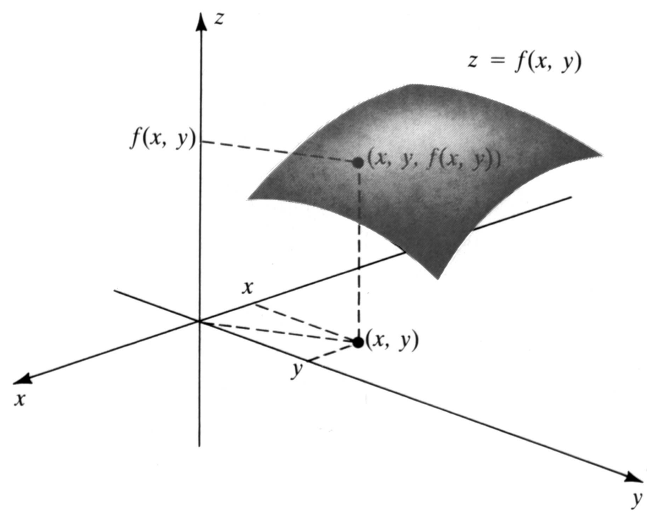
Definition 1 *A* FUNCTION *f of the two variables x and y is a rule $z = f(x,y)$ that assigns a number denoted $f(x,y)$, to each point (x,y) in a portion or all of the xy-plane. $f(x,y)$ is the* VALUE *of the function f at (x,y), and the set of points where the function is defined is called its* DOMAIN. *The* RANGE *of the function is the set of its values $f(x,y)$ for all (x,y) in its domain.*

If a function $z = f(x,y)$ is given by a formula, we assume that its domain consists of all points (x,y) for which the formula makes sense, unless a different domain is specified.

Example 1 **(a)** What is the domain of $f(x,y) = x^2 + y^2$? **(b)** What are the values $f(2,3)$ and $f(-2,-3)$ of this function at $(2,3)$ and $(-2,-3)$? **(c)** What is its range?

Answer: **(a)** The domain of f is the entire xy-plane. **(b)** $f(2,3) = 13$ • $f(-2,-3) = 13$. **(c)** The range of f is the closed infinite interval $[0,\infty)$.

Definition 2 *The graph of $z = f(x,y)$ is the surface $z = f(x,y)$ formed by the points (x,y,z) in xyz-space with (x,y) in the domain of the function and $z = f(x,y)$ (Figure 1).*



$z = f(x,y)$

FIGURE 1

†Lecture notes to accompany Section 14.1, Part 1 of *Calculus, Early Transcendentals* by Rogawski.

Fixing x *or* y*: vertical cross sections of graphs*

One way to study the graph $z = f(x, y)$ of a function of two variables is to study the graphs of the functions of one variable that are obtained by holding x or y constant.

Example 2 Determine the shape of the surface $z = x^2 + y^2$ by studying its cross sections in the planes $x = c$ perpendicular to the x-axis.

Answer: The intersection of the surface $z = x^2 + y^2$ with the plane $x = c$ is a parabola that opens upward and whose vertex is at the origin if $c = 0$ and is c^2 units above the xy-plane if $c \neq 0$ • Figure A2a • The surface has the bowl-like shape in Figure A2b

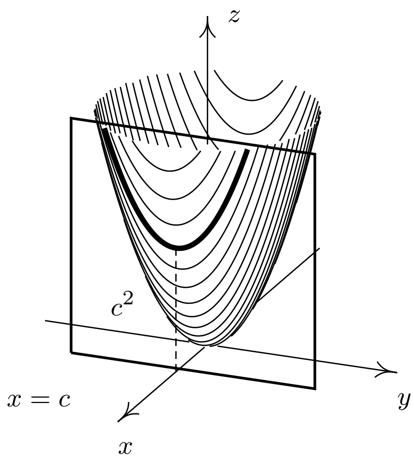


Figure A2a

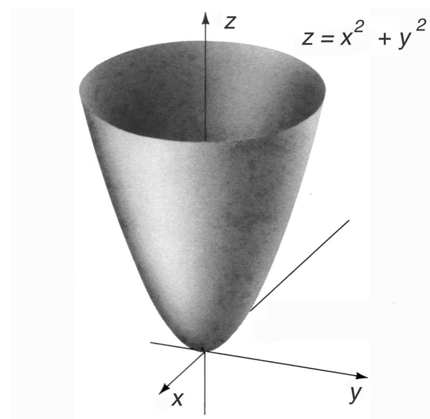


Figure A2b

Example 3 Determine the shape of the surface $z = x^2 + y^2$ of Example 2 by studyng its cross sections in the planes $y = c$ perpendicular to the y-axis.

Answer: The intersection of the surface $z = x^2 + y^2$ with the plane $y = c$ is parabola that opens upward and whose vertex is at the origin if $c = 0$ and is c^2 units above the xy-plane if $c \neq 0$. • Figure A3a • The surface has the bowl-like shape from Example 2. (Figure A3b shows the cross sections from Examples 2 and 3 together.)

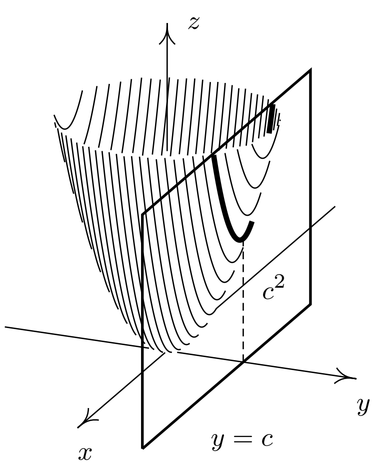


Figure A3a

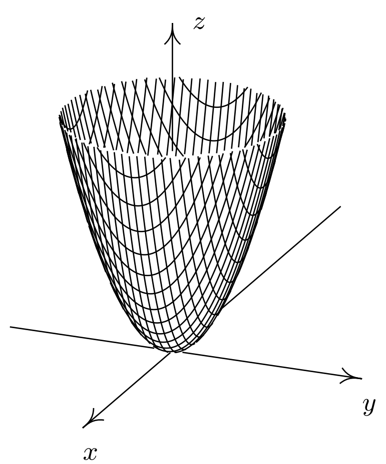


Figure A3b

Example 4 Determine the shape of the surface $z = y^2 - x^2$ by studying its cross sections in the planes $x = c$ perpendicular to the x-axis.

Answer: The intersection of the surface $z = y^2 - x^2$ with the plane $x = c$ is a parabola that opens upward and whose vertex is c^2 units below the xz-plane. • Figure A4a • The vertex is at the origin for $c = 0$ and drops below the xy-plane as c moves away from zero. • The surface has the saddle shape in Figure A4b.

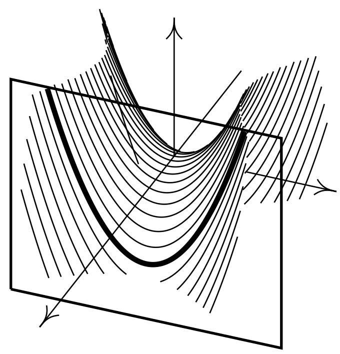


Figure A4a

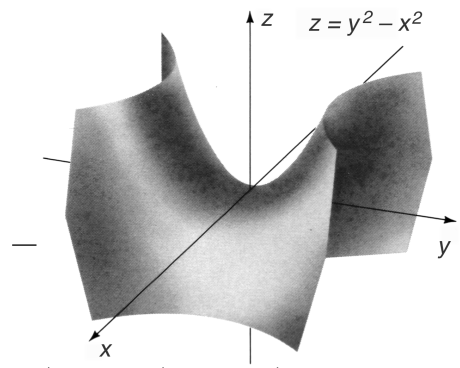


Figure A4b

Example 5 Determine the shape of the surface $z = y^2 - x^2$ of Example 4 by studying its cross sections in the planes $y = c$ perpendicular to the y-axis.

Answer: The intersection of the surface $z = y^2 - x^2$ with the plane $y = c$ is a parabola that opens downward and whose vertex is c^2 units above the xy-plane. • Figure A5a • The vertex is at the origin for $c = 0$ and rises above the xy-plane as c moves away from zero. • The surface has the saddle shape from Example 4. (Figure A5b shows the two sets of cross sections together.)

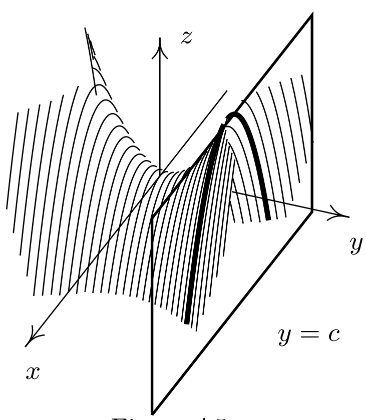


Figure A5a

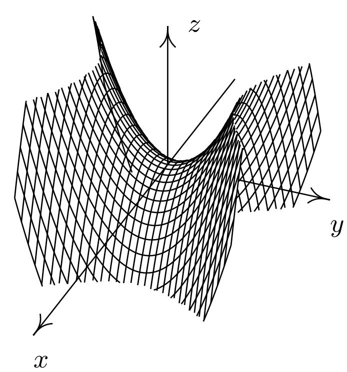


Figure A5b

Example 6 Use the curve $z = y - \frac{1}{12}y^3$ in the yz-plane of Figure 2 to determine the shape of the surface $z = y - \frac{1}{12}y^3 - \frac{1}{4}x^2$.

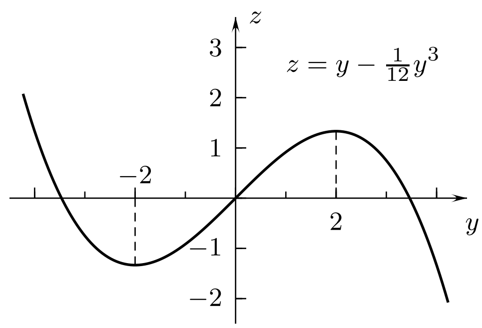


FIGURE 2

Answer: One solution: The cross section of the surface in the plane $x = c$ has the shape of the curve in Figure 2 if $c = 0$, is that curve moved down and forward if $c > 0$ and is that curve moved down and back if $c < 0$. • The surface has the boot-like shape in Figure A6
Another solution: The cross section in the plane $y = c$ is a parabola that opens downward and has its vertex on the curve in Figure 2. • The surface has the boot-like shape in Figure A6.

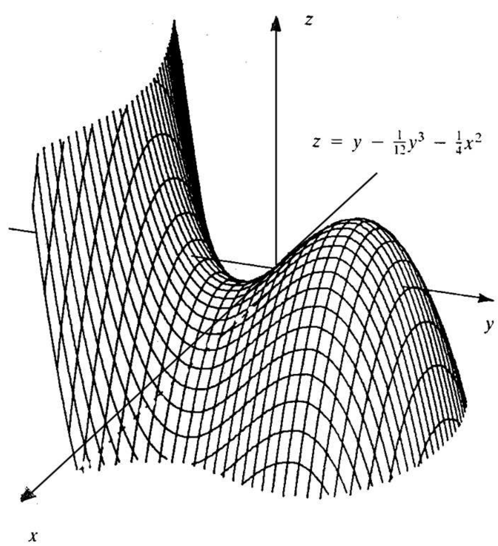


Figure A6

Horizontal cross sections and level curves

Definition 3 *The level curves (contour curves) of $z = f(x, y)$ are the curves in the xy-plane where the function is constant. They have the equations $f(x, y) = c$ with constants c (Figure 3).*

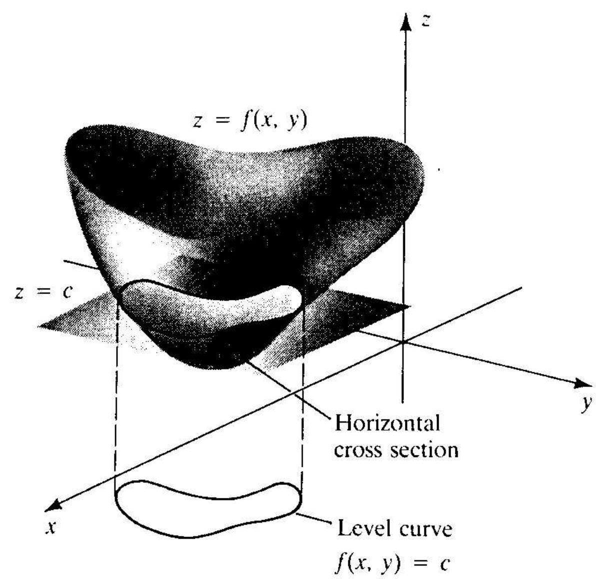


FIGURE 3

Example 7 Describe the level curves of the function $f(x, y) = x^2 + y^2$ from Examples 2 and 3.

Answer: Figure A7a shows horizontal cross sections of the graph of f and Figure A7b shows the corresponding level curves. • The level curve $f = c$ is the circle of radius $\sqrt{c}$ with its center at the origin if $c > 0$, is the origin if $x = 0$, and is empty if $c < 0$. (The surface is called a "circular paraboloid.")

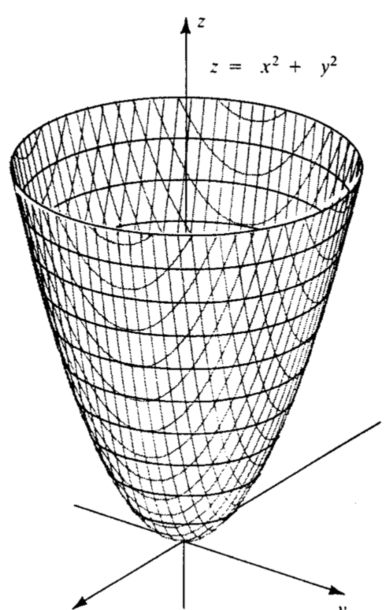


Figure A7a

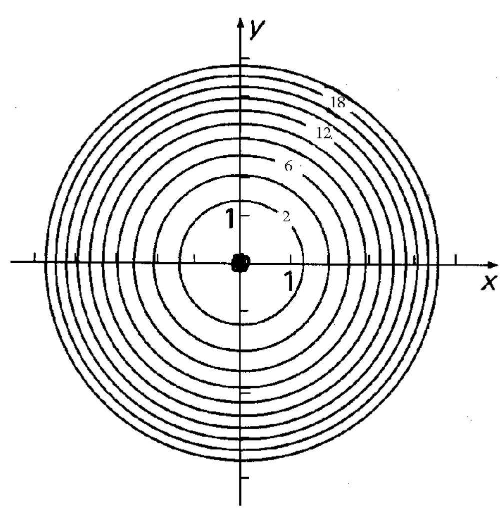


Figure A7b

Example 8 Describe the level curves of $g(x,y) = y^2 - x^2$ from Examples 4 and 5.

Answer: Figures A8a and A8b • The level curves $g = c$ is a hyperbola with the equation $y^2 - x^2 = c$. (The surface is a "hyperbolic paraboloid.")

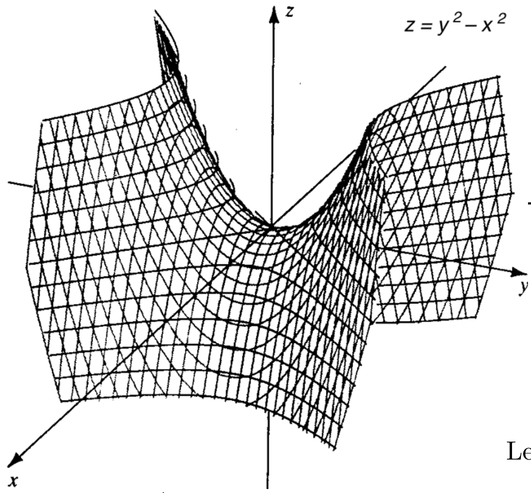
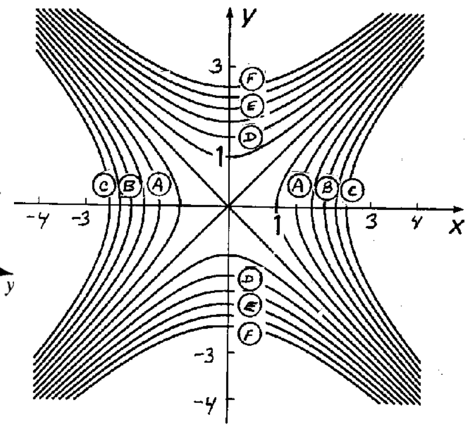


Figure A8a



Level curves of $g(x,y) = y^2 - x^2$

Figure A8b

Example 9 Figures 4 and 5 show horizontal cross sections of the graph of $h(x,y) = y - \frac{1}{12}y^3 - \frac{1}{4}x^2$ from Example 6 and the corresponding level curves of the function. Describe how the surface can be reconstructed from the level curves.

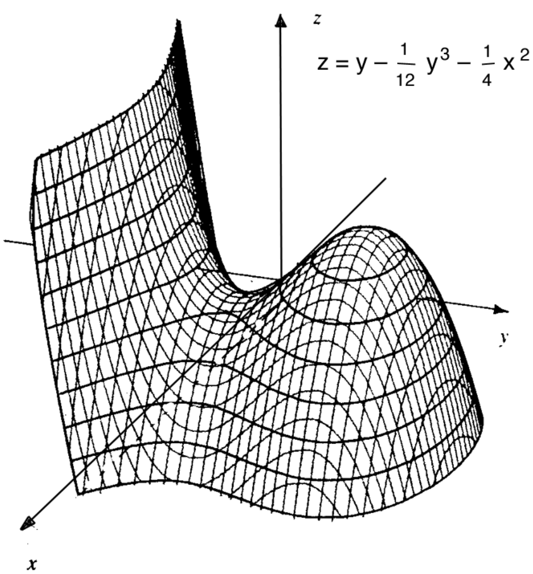


FIGURE 4

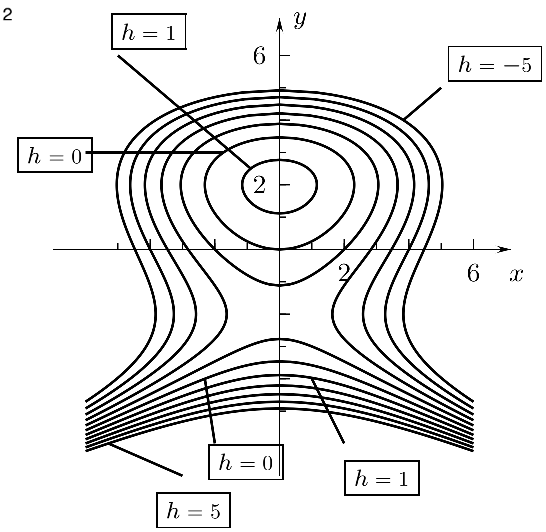


FIGURE 5

Answer: Leave the two parts of the level curve $h = 0$ on the xyplane. • Raise the two parts of the curve labeled $h = 1$ one unit on the "toe" and "leg" of the "boot." • Lower the curves above the upper part of $h = 1$ to form the sides of the "boot." • Raise the curves below the lower part of $h = 1$ to form the more of the "leg" of the "boot."

Rotating axes

The surfaces $z = kxy$ with nonzero constants k are important in the study of maxima and minima of functions with two variables. Their shapes can be determined by introducing new $x'y'$-coordinates by rotating the x- and y-axes 45° counterclockwise as in Figure 4.

$$x = \frac{1}{\sqrt{2}}(x' - y'),\ y = \frac{1}{\sqrt{2}}(x' + y'). \qquad \textbf{(1)}$$

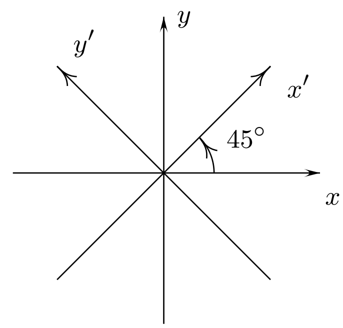


FIGURE 4

Example 10 Use $x'y'$-coordinates as in Figure 4 to analyze the surface $z = -2xy$.

Answer: The graph is the surface $z = y^2 - x^2$ of Figure A4b rotated 45° as in Figure A10. (Notice that the x- and y-axes are on the surface.)

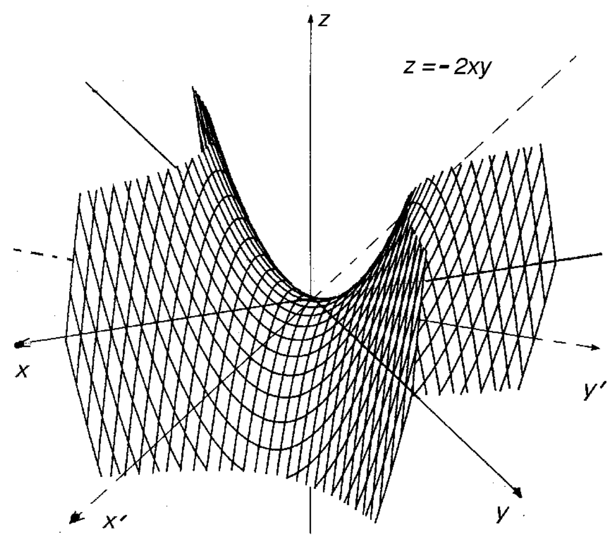


Figure A10

Interactive Examples

Work the following Interactive Examples on Shenk's web page, http//www.math.ucsd.edu/˜ashenk/:[†]

Section 14.1: Examples 1–6

[†]The chapter and section numbers on Shenk's web site refer to his calculus manuscript and not to the chapters and sections of the textbook for the course.

Exercises

1. Add axes to Figure 5 so it is the graph of $f(x, y) = y^2$.

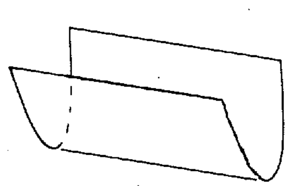


FIGURE 5

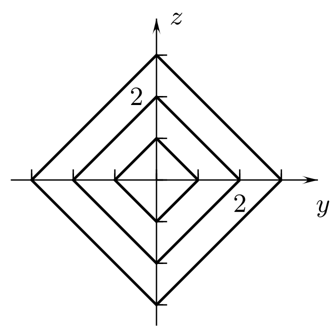


FIGURE 6

2. What are the values of $L(x, y) = |x| + |y|$ on its three level curves in Figure 6?

3. Draw the graph of the function $z = -\sqrt{1 - x^2 - y^2}$.

4. Draw and label the level curves of $N(x, y) = y - \frac{1}{2}x$ where it has the values $c = 0, \pm 1$, and ± 2.

5. Draw and label the level curves of $S(x, y) = y - \sin x$ where it has the values $0, \pm 2, \pm 4$.

6. **(a)** Explain why the horizontal cross sections of $z = \ln(x^2 + y^2)$ and of $z = \dfrac{1}{\sqrt{x^2 + y^2}}$ are circles.

(b) Match the surfaces in Figures 7 and 8 to their equations in part (a). Explain your choices.

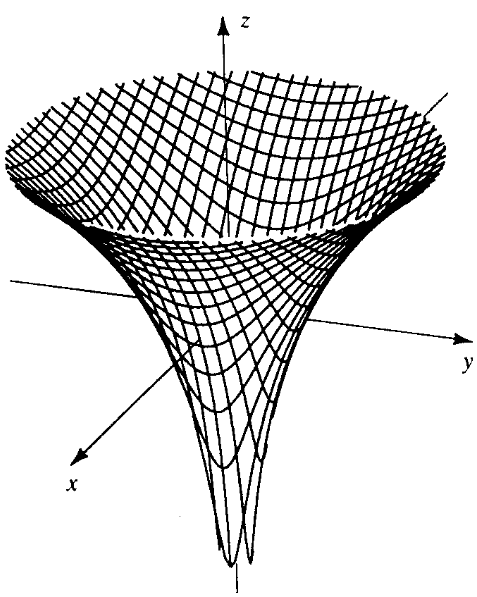


FIGURE 7

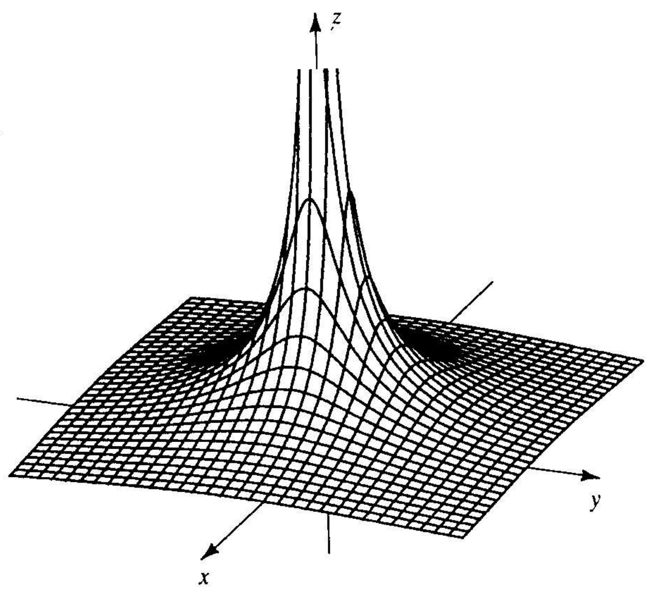


FIGURE 8

7. Match the functions **(a)** $z = \sin y$, **(b)** $z = -\sin x \sin y$, **(c)** $z = \sin^2 y + \frac{1}{2}x^2$, and **(d)** $z = 3e^{-x/5} \sin y$ to their graphs in Figures 9 through 12.

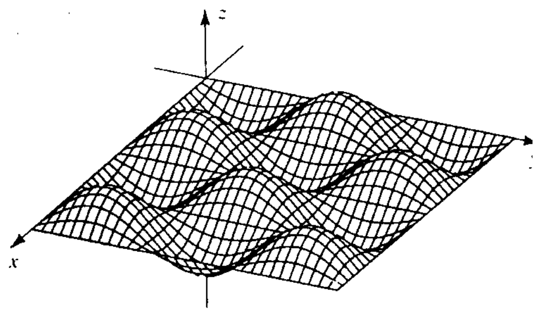


FIGURE 9

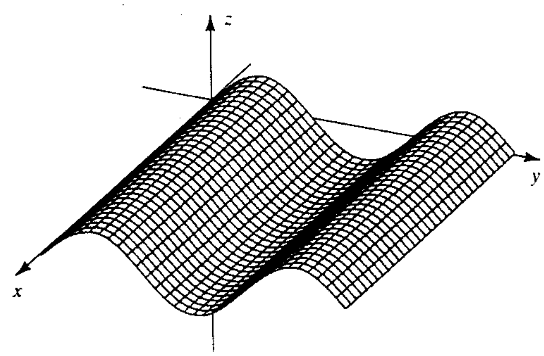


FIGURE 10

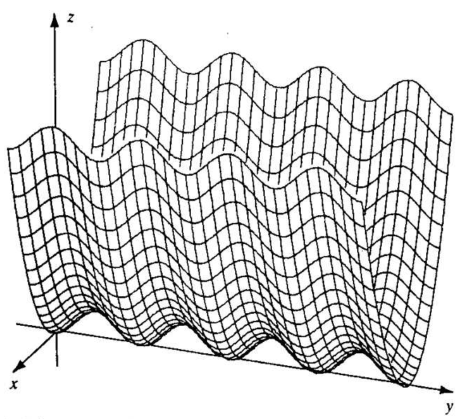


FIGURE 11

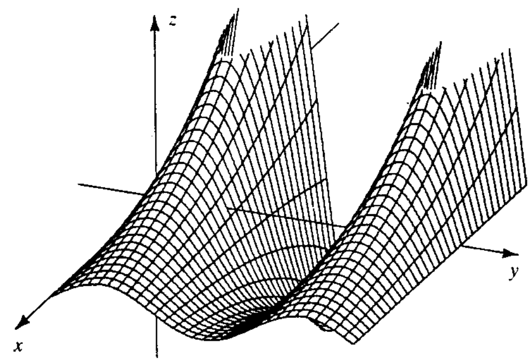


FIGURE 12

8. Match the functions of Exercise 7 with their level curves in Figures 13 through 16.

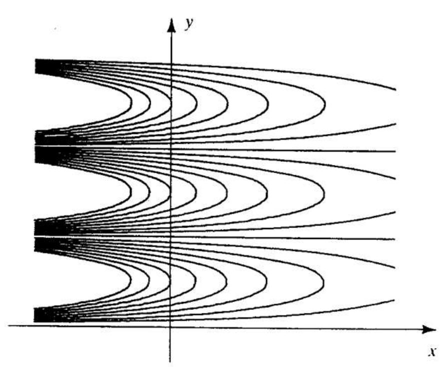


FIGURE 13

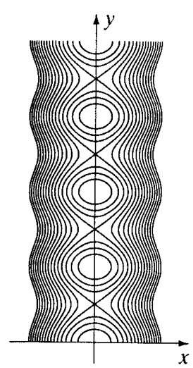


FIGURE 14

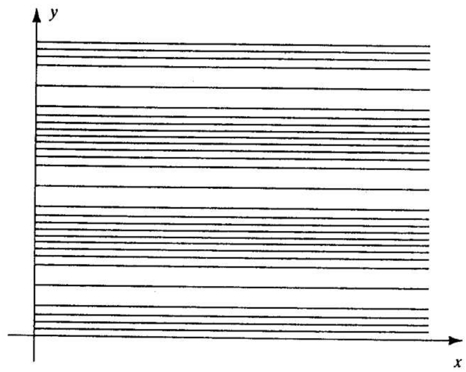


FIGURE 15

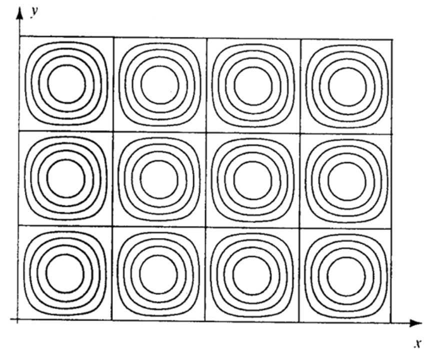


FIGURE 16

Selected answers

1. Figure A1

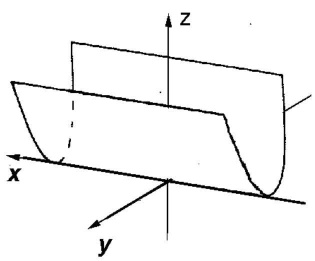


Figure A1

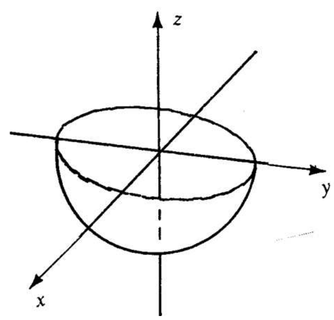


Figure A3

2. $L = 3$ on the outer square • $L = 2$ on the middle square • $L = 1$ on the inner square

3. Figure A3

4. Figure A4

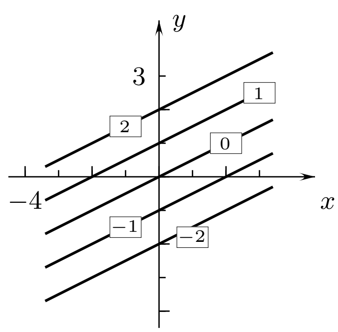


Figure A4