(8/19/08)

# Math 20C. Lecture Examples.

## Sections 14.2 and 14.3. Limits and partial derivatives†

***Example 1*** What is $\lim\limits_{(x,y)\to(3,2)}(x^2+y^2)$?

**Answer:** $\lim\limits_{(x,y)\to(3,2)}(x^2+y^2)=13$

***Example 2*** Find the $x$- and $y$-derivatives of $f(x,y)=x^3y-x^2y^5+x$.

**Answer:** $\dfrac{\partial f}{\partial x}=3x^2y-2xy^5+1$ • $\dfrac{\partial f}{\partial y}=x^3-5x^2y^4$

***Example 3*** What are $g_x(2,5)$ and $g_y(2,5)$ for $g(x,y)=x^2e^{3y}$?

**Answer:** $g_x(2,5)=4e^{15}$ • $g_y(2,5)=12e^{15}$

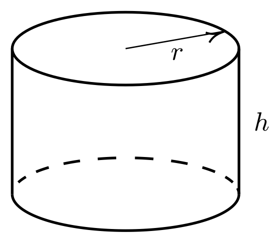
***Example 4*** The volume of a right circular cylinder of radius $r$ and height $h$ is equal to the product $V(r,h)=\pi r^2h$ of its height $h$ and the area $\pi r^2$ of its base (Figure 1). What are **(a)** the rate of change of the volume with respect to the radius and **(b)** the rate of change of the volume with respect to the height and what are their geometric significance?

[Area of base ] $=\pi r^2$

[Volume] $=\pi r^2h$

[Lateral surface area] $=2\pi rh$

FIGURE 1




**Answer: (a)** $\dfrac{\partial V}{\partial r}=2\pi rh$ is the area of the lateral surface (the sides) of the cylinder.

**(b)** $\dfrac{\partial V}{\partial h}=\pi r^2$ is the area of the base.

†Lecture notes to accompany Sections 14.2 and 14.3 of *Calculus, Early Transcendentals* by Rogawski.

***Partial derivatives as slopes of tangent lines***

When we hold $y$ equal to a constant $y = y_0$, $z = f(x, y)$ becomes the function $z = f(x, y_0)$ of $x$, whose graph is the intersection of the surface $z = f(x, y)$ with the vertical plane $y = y_0$ (Figure 2). The $x$-derivative $f_x(x_0, y_0)$ is the slope in the positive $x$-direction of the tangent line to this curve at $x = x_0$.

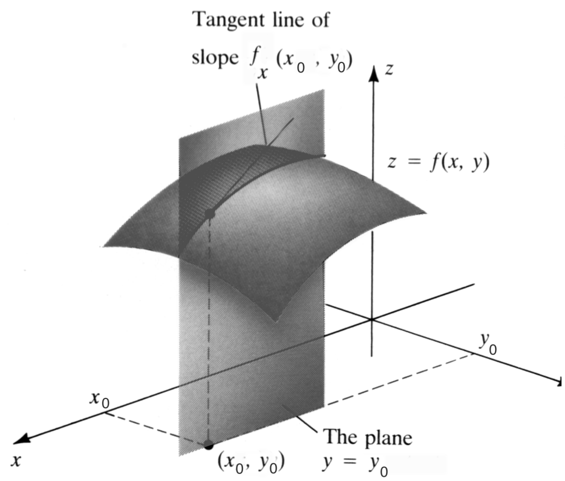



FIGURE 2

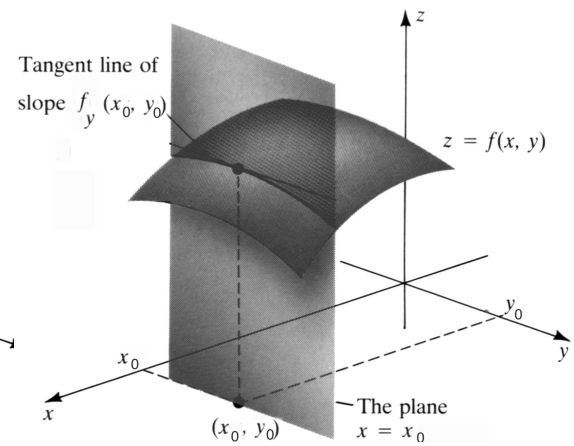



FIGURE 3

Similarly, when we hold $x$ equal to a constant $x_0$, $z = f(x, y)$ becomes the function $z = f(x_0, y)$ of $y$, whose graph is the intersection of the surface with the plane $x = x_0$ (Figure 3), and the $y$-derivative $f_y(x_0, y_0)$ is the slope in the positive $y$-direction of the tangent line to this curve at $y = y_0$.

***Example 5*** **(a)** Is the $x$-derivative $f_x(x_0, y_0)$ in Figure 2 positive or negative? **(b)** The tangent line in Figure 3 is approximately parallel to the $y$-axis. What does this imply about the $y$-derivative $f_y(x_0, y_0)$?

**Answer: (a)** $f_x(x_0, y_0)$ is negative. **(b)** The tangent line is approximately horizontal so the derivative is approximately zero.

***Example 6*** The table below is from a study of the effect of exercise on the blood pressure of women. $P = P(t, E)$ is the average blood pressure, measured in millimeters of mercury (mm Hg), of women of age $t$ years who are exercising at the rate of $E$ watts.[1] (One watt is 0.86 Calories per hour.) What is the approximate rate of change with respect to age of the average blood pressure of forty-five-year old women who are exercising at the rate of 100 watts?

$P = P(t, E)$ (millimeters of mercury)

| | $t = 25$ | $t = 35$ | $t = 45$ | $t = 55$ | $t = 65$ |
|---|---|---|---|---|---|
| $E = 150$ | 178 | 180 | 197 | 209 | 195 |
| $E = 100$ | 163 | 165 | 181 | 199 | 200 |
| $E = 50$ | 145 | 149 | 167 | 177 | 181 |
| $E = 0$ | 122 | 125 | 132 | 140 | 158 |

(1) Data adapted from *Geigy Scientific Tables*, edited by C. Lentner, Vol. 5, Basel, Switzerland: CIBA-GEIGY Limited, 1990, p. 29.

**Answer:** $P_t(45,100) \approx 1.8$ millimeters of mercury per year (using a right difference quotient); or
$P_t(45,100) \approx 1.6$ millimeters of mercury per year (using a left difference quotient); or
$P_t(45,100) \approx 1.7$ millimeters of mercury per year (using a centered difference quotient)

***Example 7*** Use the table from Example 6 to fine the approximate rate of change with respect to exercise rate ~~age~~ of the average blood pressure of fifty-five-year old women who are exercising at the rate of 75 watts.

$P = P(t,E)$ (millimeters of mercury)

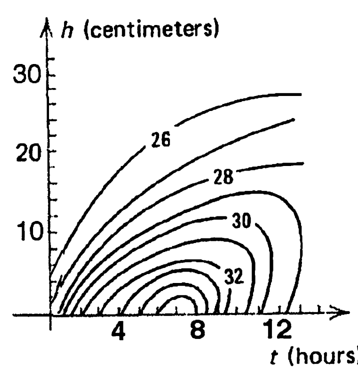
| | $t=25$ | $t=35$ | $t=45$ | $t=55$ | $t=65$ |
|---|---|---|---|---|---|
| $E=150$ | 178 | 180 | 197 | 209 | 195 |
| $E=100$ | 163 | 165 | 181 | 199 | 200 |
| $E=50$ | 145 | 149 | 167 | 177 | 181 |
| $E=0$ | 122 | 125 | 132 | 140 | 158 |

**Answer:** $\left.\dfrac{\partial P}{\partial E}\right|_{(62,75)} \approx 0.44$ millimeters of mercury per watt

## *Estimating partial derivatives from level curves*

***Example 8*** Figure 4 shows level curves of the temperature $T = T(t,h)$ (degrees Fahrenheit) as a function of time $t$ (hours) and the depth $h$ (centimeters) beneath the surface of the ground at O'Neil, Nebraska, from noon one day ($t = 0$) until the next morning.[2] **(a)** What was the approximate rate of change of the temperature with respect to time at 4:00 PM at a point 14 centimeters beneath the surface of the ground? **(b)** What was the approximate rate of change of the temperature with respect to depth at 4:00 PM at a point 14 centimeters beneath the surface of the ground?

FIGURE 4




[2] Data adapted from *Fundamentals of Air Pollution* by S. Williamson, Reading, MA: Addison Wesley, 1973.

**Answer:** **(a)** Figure A8 • $T_t(4,14) \approx 0.5$ degree per hour **(b)** $T_h(4,14) \approx$ [redacted] degree per centimeter

-1.7

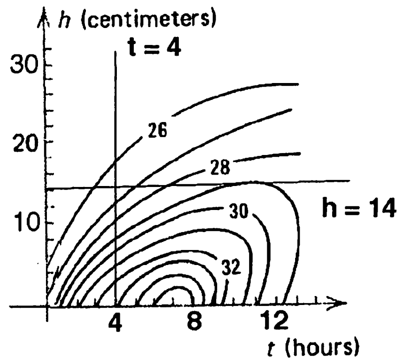



Figure A8

***Example 9*** What are the first-order partial derivatives of $f = x^2y^3z^4$?

**Answer:** $f_x = 2xy^3z^4$ • $f_y = 3x^2y^2z^4$ • $f_z = 4x^2y^3z^3$

***Example 10*** What are **(a)** $h_{yz}$ and **(b)** $h_{zy}$ for $h(x,y,z) = e^x \sin y \cos z$?

**Answer:** **(a)** $h_{yz} = -e^x \cos y \sin z$ **(b)** $h_{zy} = -e^x \cos y \sin z$

***Example 10*** Find the fourth derivative $\dfrac{\partial^4}{\partial w \partial x \partial y \partial z}(w^2x^2y^2z^2)$.

**Answer:** $\dfrac{\partial^4}{\partial w \partial x \partial y \partial z}(w^2x^2y^2z^2) = 16wxyz$

## Interactive Examples

Work the following Interactive Examples on Shenk's web page, http//www.math.ucsd.edu/~ashenk/:†

Section 14.3: Examples 1 through 5

Section 14.7: Examlple 2

Section 14.8: Example 2