(8/17/10)

Math 20C. Lecture Examples.

## Section 14.4. Linear approximations and tangent planes[†]

A function $z = f(x, y)$ of two variables is LINEAR if its graph in xyz-space is a plane. Equations of planes were found in Section 12.5 by using their normal vectors. Here we will need instead equations given in the next theorem for planes in terms of the slopes of their cross sections in the x- and y-directions.

Theorem 1(a) (The slope-intercept equation of a plane)   *Suppose that the z-intercept of a plane is b, the slope of its vertical cross sections in the positive x-direction is m_1, and the slope of its vertical cross sections in the positive y-direction is m_2 (Figure 1). Then the plane has the equation*
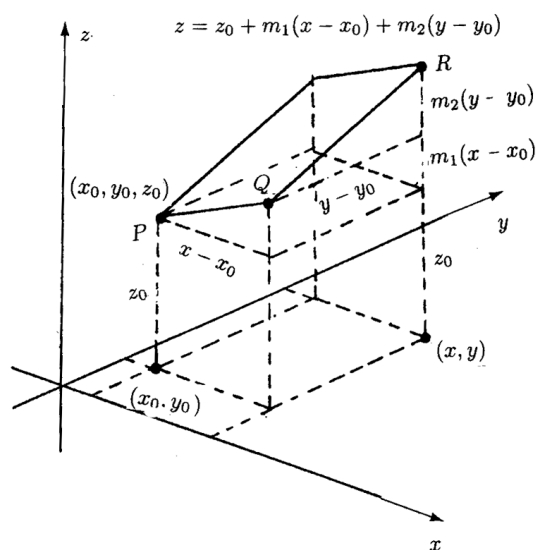
$$z = m_1 x + m_2 y + b. \tag{1}$$

(b) (The point-slope equation of a plane)   *Suppose that a plane contains the point (x_0, y_0, z_0), the slope of its vertical cross sections in the positive x-direction is m_1, and the slope of its vertical cross sections in the positive y-direction is m_2 (Figure 4). Then the plane has the equation*

$$z = z_0 + m_1(x - x_0) + m_2(y - y_0). \tag{2}$$
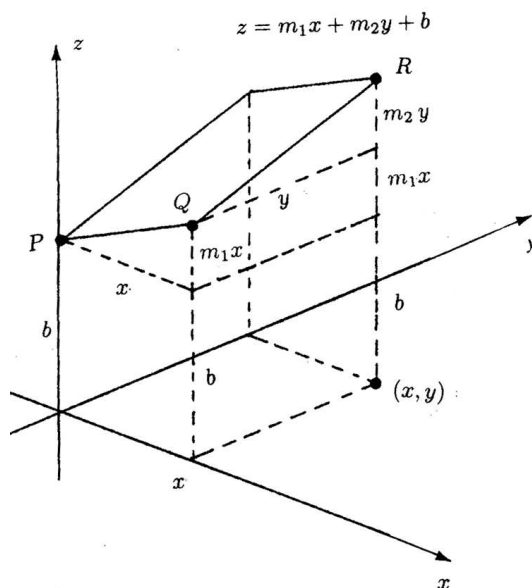
The slope-intercept equation

FIGURE 1

The point-slope equation

FIGURE 2

[†]Lecture notes to accompany Section 14.4 of *Calculus, Early Transcendentals* by Rogawski.

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Example 1   Give an equation of the plane with slope -6 in the positive x-direction, slope 7 in the positive y-direction, and z-intercept 10.

Answer: $z = -6x + 7y + 10$

Example 2   Give an equation of the plane through the point $(1, 2, 3)$ with slope 4 in the positive x-direction and slope -5 in the positive y-direction.

Answer: $z = 3 + 4(x - 1) - 5(y - 2)$

Example 3   Find a formula for the linear function $z = g(x, y)$ whose values are given in the following table.

Values of $z = g(x, y)$

	$x = -3$	$x = 0$	$x = 3$
$y = 2$	8	14	20
$y = 0$	14	20	26
$y = -2$	20	26	32

Answer: $g(x, y) = 2x - 3y + 20.$

Example 4   Find a formula for the linear function $z = h(x, y)$ whose level curves are given in Figure 3.
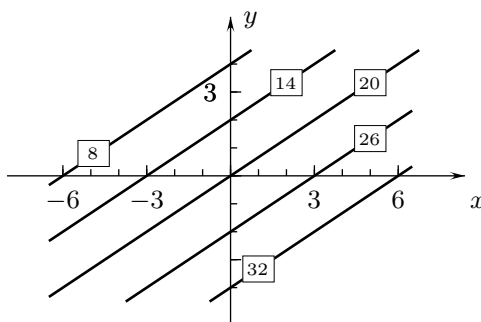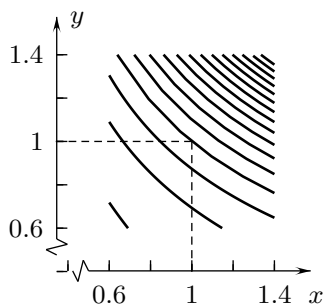

FIGURE 3

Zooming in on level curves of a nonlinear $\mathbf{z} = \mathbf{f(x, y)}$

If a function $y = f(x)$ of one variable has a derivative at x_0 and the graph $y = f(x)$ is generated by a calculator or computer in a small enough window containing the point $(x_0, f(x_0))$, the displayed portion of the graph will look like a line. This occurs because the graph is closely approximated by the tangent line near that point.

Graphs of functions of two variables with continuous first derivatives are closely approximated by planes when viewed in small windows. Consequently, their level curves in small windows look like parallel lines. This is illustrated by the level curves of $K(x, y) = 3x^2 y^3 + x$ in Figures 4 through 6. The level curves look more like equally spaced parallel lines in Figure 5 than Figure 4, and even more like equally spaced parallel lines in Figure 6.
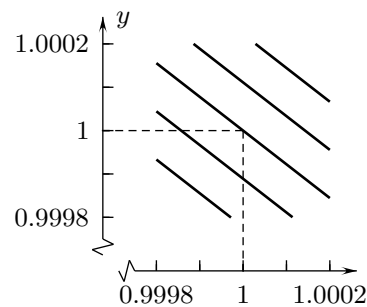
$K = 1, 2, 3, \ldots, 17$

FIGURE 4
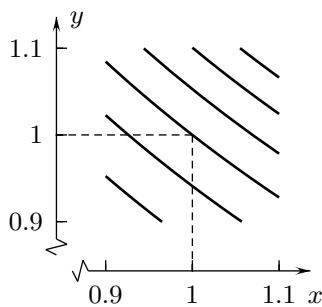
$K = 3, 3.5, \ldots, 5.5$

FIGURE 5

$K = 3.998, \ldots, 4.002$

FIGURE 6

Example 5   **(a) The function $K(x, y) = 3x^2 y^3 + x$ has the value 4 at the point $P = (1, 1)$ in Figure 7 and the value 3.999 at the point Q. What is the approximate value of $K_x(1, 1)$?**

(b) K has also has the value 3.999 at the point R in Figure 8. What is the approximate value of $K_y(1, 1)$?   (c) Find the exact derivatives in parts (a) and (b) by using the formula $K(x, y) = 3x^2 y^3 + x$.
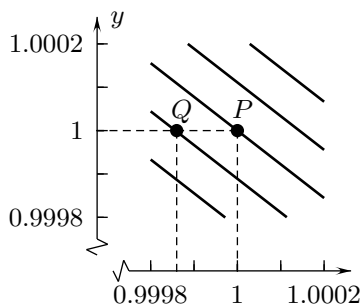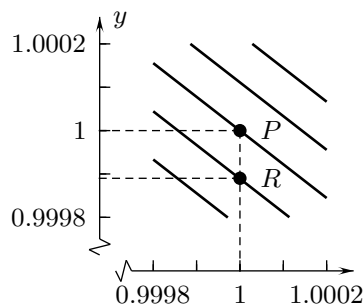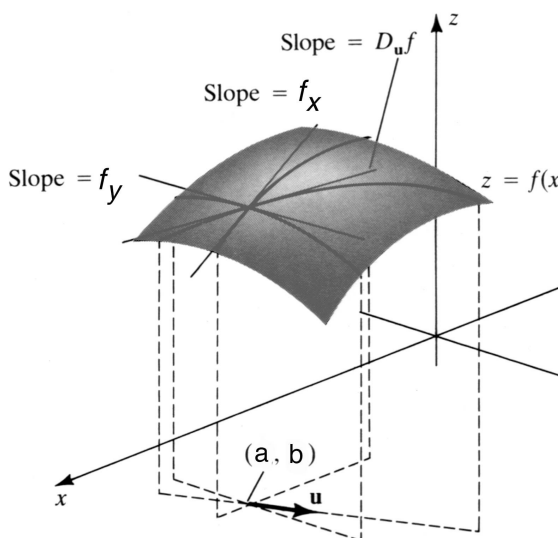

FIGURE 7

FIGURE 8

Answer: (a) $K_x(1, 1) \approx \dfrac{0.001}{0.00014} \doteq 7.14$ **(b)** $K_y(1, 1) \approx \dfrac{0.001}{0.00011} \doteq 9$ **(c)** $K_x(1, 1) = 7$ ● $K_y = 9$

Equations of tangent planes

If $z = f(x, y)$ has continuous first derivatives in an open circle centered at (a, b), then the linear function
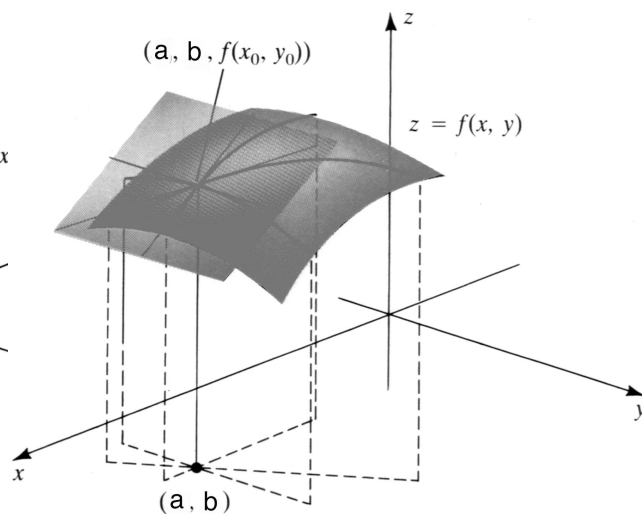
$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

has the same value and the same first derivatives at (a, b) as $z = f(x, y)$. Consequently, it has the same directional derivatives in all directions at (a, b) as $z = f(x, y)$ (see Section 14.5). This implies that all of the tangent lines to vertical cross sections of the graph of f through $x = a, y = b$ (Figure 9) are in the plane that is the graph of L (Figure 10). That plane is the tangent plane to the graph of f at $x = a, y = b$.

Tangent lines to vertical cross sections

FIGURE 9

The tangent plane

FIGURE 10

Definition 1   If $z = f(x, y)$ has continuous first derivatives in an open circle centered at (x_0, y_0), then the tangent plane to its graph at $x = a, y = b$ is

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

Example 6   Give an equation of the tangent plane to the graph of $f(x, y) = x^3 y^4$ at $x = 1, y = 2$.

   Answer: Tangent plane: $z = 16 + 48(x - 1) + 32(y - 2)$

Estimating errors

Example 7   **The volume of a cube is determined by measuring its width. The width is measured to be 2 centimeters with an error ≤ 0.02 centimeters. Use a tangent-line approximation to estimate the maximum possible error in the calculated volume.**

Answer: $V = w^3$ ● Tangent line at $w = 2$: $V = 8 + 12(w - 2)$ ● Suppose w is the exact width. ● [Error] $= |V(w) - 8| \approx |12(w - 2)| \leq 12(0.02) = 0.12$ cubic centimeters

Example 8   **The radius of a right circular cylinder is measured to be $r = 10 \pm 0.01$ centimeters (meaning that it is measured to be 10 centimeters with an error ≤ 0.01) and its height is measured to be $h = 15 \pm 0.005$ centimeters (meaning that it is measured to be 15 centimeters with an error ≤ 0.005). Use a tangent-plane approximation to estimate the maximum possible error in the calculated volume of the cylinder.**

Answer: $V(r, h) = \pi r^2 h$ ● Tangent plane at $r = 10, h = 15$: $V = 1500\pi + 300\pi(r - 10) + 100\pi(h - 15)$ ● Suppose that r and h are the exact radius and height. ● [Error] $= |V(r, h) - 1500\pi| \approx |300\pi(r - 10) + 100\pi(h - 15)| \leq 300\pi(0.01) + 100\pi(0.005) = 3.5\pi$ cubic centimeters ● The maximum possible error is approximately $3.5\pi \doteq 11$ cubic centimeters.

Interactive Examples

Work the following Interactive Examples on Shenk's web page, http//www.math.ucsd.edu/˜ashenk/:[‡]

Section 14.6: Examples 1 through 7

[‡]The chapter and section numbers on Shenk's web site refer to his calculus manuscript and not to the chapters and sections of the textbook for the course.