(8/17/08)

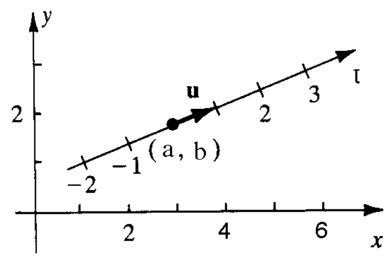
Math 20C. Lecture Examples.

Section 14.5, Part 1. Directional derivatives and gradient vectors in the plane†

The x-derivative $f_x(a,b)$ is the derivative of f at (a,b) in the direction of the unit vector $\mathbf{u}$, and the y-derivative $f_y(a,b)$ is the derivative in the direction of the unit vector $\mathbf{j}$. To find the derivative of $z = f(x,y)$ at (a,b) in the direction of an arbitrary unit vector $\mathbf{u} = \langle u_1, u_2\rangle$, we introduce an t-axis, as in Figure 1, with its origin at (a,b), with its positive direction in the direction of $\mathbf{u}$, and with the scale used on the x- and y-axes. Then the point at t on the t-axis has xy-coordinates $x = a + tu_1, y = b + tu_2$, and the value of $z = f(x,y)$ at the point t on the t-axis is

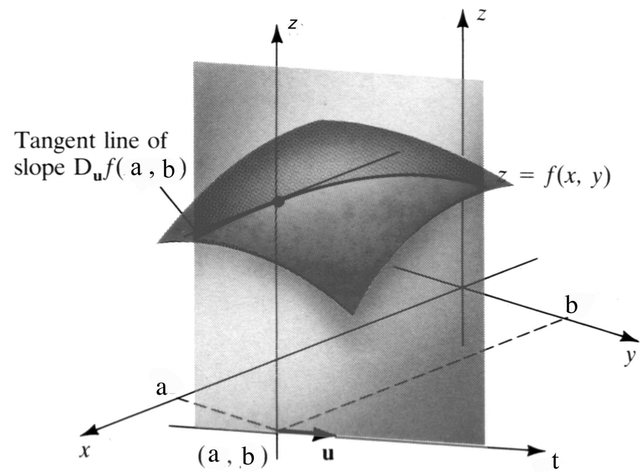
$$F(t) = f(a + tu_1, b + tu_2). \tag{1}$$

We call $z = F(t)$ the CROSS SECTION through (a,b) of $z = f(x,y)$ in the direction of $\mathbf{u}$. Its t-derivative at $t = 0$ is the directional derivative of $f$at (a,b).



$$\begin{cases} x &= a + tu_1 \\ y &= b + tu_2 \end{cases}$$

FIGURE 1



Tangent line of slope

$F'(0) = D_{\mathbf{u}}f(a,b)$

FIGURE 2

Definition 1 *The directional derivative of $z = f(x,y)$ at (a,b) in the direction of the unit vector $\mathbf{u} = \langle u_1, u_2\rangle$ is the derivative of the cross section function* **(1)** *at $t = 0$:*

$$D_{\mathbf{u}}f(a,b) = \left[\frac{d}{dt}f(a + tu_1, b + tu_2)\right]_{t=0}. \tag{2}$$

The directional derivative **(2)** is the rate of change of f at (a,b) in the direction of $\mathbf{u}$. Its geometric meaning is shown in Figure 2. We introduce a second vertical z-axis with its origin at (a,b) as in Figure 2. Then the graph of $z = F(t)$ is the intersection of the surface $z = f(x,y)$ with the tz-plane and the directional derivative of $z = f(x,y)$ is the slope of the tangent line to this curve in the positive t-direction at $t = 0$.

† Lecture notes to accompany Section 14.5, Part 1 of *Calculus, Early Transcendentals* by Rogawski.

The next theorem is used to calculate directional derivatives from partial derivatives.

Theorem 1[†] *For any unit vector* $\mathbf{u} = \langle u_1, u_2\rangle$*, the (directional) derivative of* $z = f(x, y)$ *at* (a, b) *in the direction of* $\mathbf{u}$ *is*

$$D_{\mathbf{u}} f(x_0, y_0) = f_x(a, b)u_1 + f_y(a, b)u_2. \quad \mathbf{(3)}$$

Remember formula **(3)** as the following statement: the directional derivative of $z = f(x, y)$ in the direction of $\mathbf{u}$ equals the x-derivative of f multiplied by the x-component of $\mathbf{u}$, plus the y-derivative of f multiplied by the y-component of $\mathbf{u}$.

Example 1 Find the directional derivative of $f(x, y) = -4xy - \frac{1}{4}x^4 - \frac{1}{4}y^4$ at $(1, -1)$ in the direction of the unit vector $\mathbf{u} = \langle \frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2}\rangle$ (Figure 3).

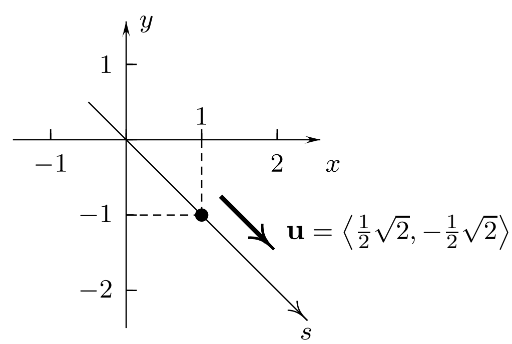


FIGURE 3

Answer: $D_{\mathbf{u}} f(1, -1) = 3\sqrt{2}$

Figures 4 and 5 show the geometric interpretation of Example 1. The line in the xy-plane through $(1, -1)$ in the direction of the unit vector $\mathbf{u} = \langle \frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2}\rangle$ has the equations

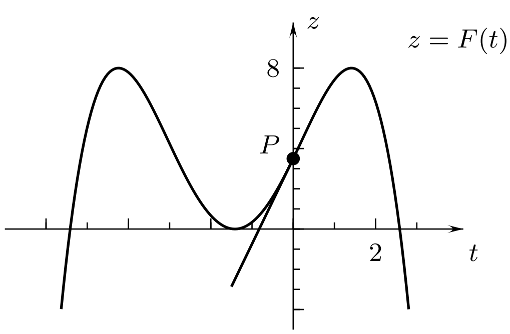
$$x = 1 + \tfrac{1}{2}\sqrt{2}\,t, y = -1 - \tfrac{1}{2}\sqrt{2}\,t$$

with distance t as parameter and $t = 0$ at $(1, -1)$. The formula $f(x, y) = -4xy - \frac{1}{4}x^4 - \frac{1}{4}y^4$ shows that the cross section of $z = f(x, y)$ through $(1, 1)$ in the direction of $\mathbf{u}$ is

$$\begin{aligned} F(t) &= -4\Big(1 + \tfrac{1}{2}\sqrt{2}\,t\Big)\Big(-1 - \tfrac{1}{2}\sqrt{2}\,t\Big) - \tfrac{1}{4}\Big(1 + \tfrac{1}{2}\sqrt{2}\,t\Big)^4 - \tfrac{1}{4}\Big(-1 - \tfrac{1}{2}\sqrt{2}\,t\Big)^4 \\ &= 4\Big(1 + \tfrac{1}{2}\sqrt{2}\,t\Big)^2 - \tfrac{1}{2}\Big(1 + \tfrac{1}{2}\sqrt{2}\,t\Big)^4. \end{aligned}$$

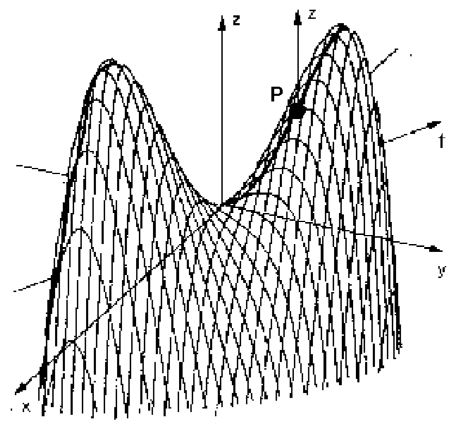
[†]We assume in this section that the functions involved have continuous first-order partial derivatives in open circles centered at all points (x, y) that are being considered.

The graph of this function is shown in the tz-plane of Figure 4. The slope of its tangent line at $t = 0$ is the directional derivative from Example 1. The corresponding cross section of the surface $z = f(x, y)$ is the curve over the t-axis drawn with a heavy line in Figure 5, and the directional derivative is the slope of this curve in the positive t-direction at the point $P = (1, -1, f(1, -1))$ on the surface.



Cross section of $z = f(x, y)$
through $(1, -1)$ in the
direction of $\mathbf{u} = \langle \frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2} \rangle$

FIGURE 4



$z = f(x, y)$

FIGURE 5

Example 2 What is the derivative of $f(x, y) = x^2y^5$ at $P = (3, 1)$ in the direction toward $Q = (4, -3)$?

Answer: $D_{\mathbf{u}}f(3, 1) = -\dfrac{174}{\sqrt{17}}$

Example 3 What is the derivative of $h(x, y) = e^{xy}$ at $(2, 3)$ in the direction at an angle of $\frac{2}{3}\pi$ radians from the positive x-direction?

Answer: Figure A3 • $\mathbf{u} = \langle -\frac{1}{2}, \frac{1}{2}\sqrt{3} \rangle$ • $D_{\mathbf{u}}h(2, 3) = (-\frac{3}{2} + \sqrt{3})e^6$

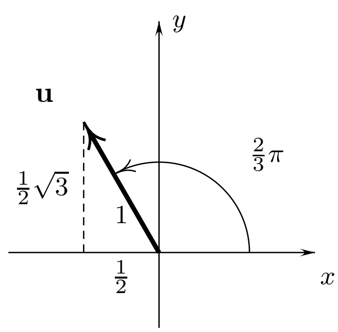


Figure A3

Example 4 Figure 6 shows level curves of the temperature $T = T(x, y)$ (degrees Celsius) of the surface of the ocean off the west coast of the United States at one time.[(1)] **(a)** Express the rate of change toward the northeast of the temperature at point P in the drawing as a directional derivative, assuming that $P = (1240, 1000)$. **(b)** Find the approximate value of this rate of change.

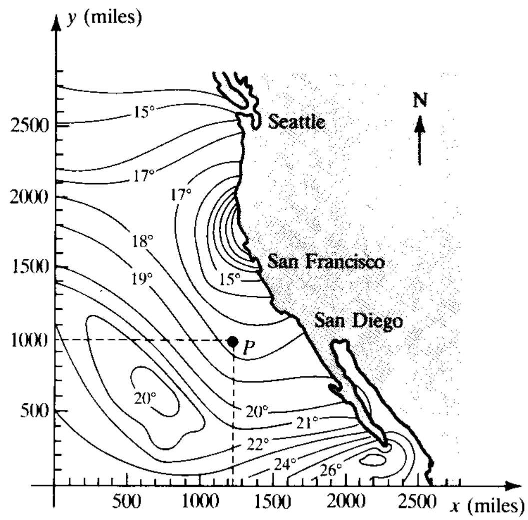


FIGURE 6

Answer: (a) The rate of change of the temperature toward the northeast at P is $D_{\mathbf{u}}T(1240, 1000)$, where $\mathbf{u}$ is the unit vector in the northeast direction. •
(b) Figure A4. • One answer: $D_{\mathbf{u}}T(1240, 1000) \approx -0.005$ degrees per mile

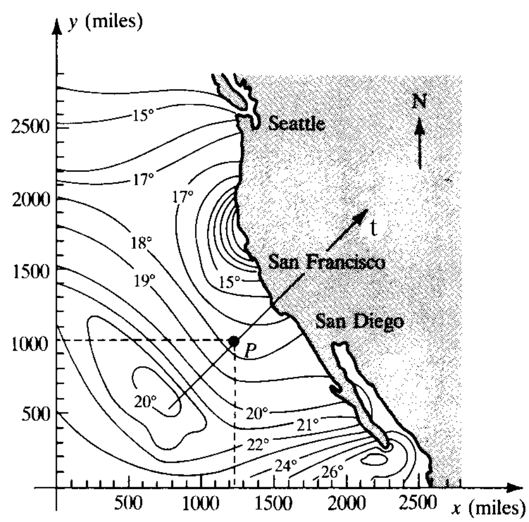


Figure A4

(1) Data adapted from *Zoogeography of the Sea* by S. Elkman, London: Sidgwich and Jackson, 1953, p. 144.

The gradient vector

The formula

$$D_{\mathbf{u}}f(a,b) = f_x(a,b)\,u_1 + f_y(a,b)\,u_2$$

from Theorem 1 for the derivative of f at (a,b) in the direction of the unit vector $\mathbf{u} = \langle u_1, u_2\rangle$ has the form of the dot product of $\mathbf{u}$ with the vector $\langle f_x, f_y\rangle$ at (a,b). This leads us to define the latter to be the GRADIENT VECTOR of f, which is denoted ∇f.†

Definition 2 *The gradient vector of $f(x,y)$ at (a,b) is*

$$\nabla f(a,b) = \langle f_x(a,b), f_y(a,b)\rangle. \quad \textbf{(4)}$$

The gradient vector **(4)** is drawn as an arrow with its base at (a,b). Because its length is a derivative (a rate of change) rather than a distance, its length can be measured with any convenient scale. The scales on the coordinate axes are used, however, whenever possible.

Example 5 Draw $\nabla f(1,1), \nabla f(-1,2)$, and $\nabla f(-2,-1)$ for $f(x,y) = x^2y$. Use the scale on the x- and y-axes to measure the lengths of the arrows.

Answer: $\nabla f(1,1) = \langle 2,1\rangle$ • $\nabla f(-1,2) = \langle -4,1\rangle$ • $\nabla f(-2,-1) = \langle 4,4\rangle$ • Figure A5

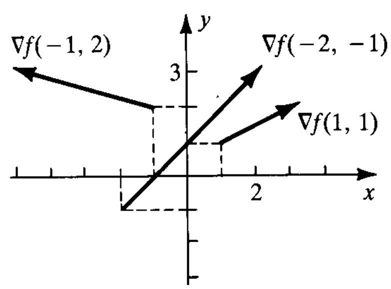


Figure A5

With Definition 2, formula **(3)** for the directional derivative becomes

$$D_{\mathbf{u}}f(a,b) = \nabla f(a,b)\cdot\mathbf{u} = |\nabla f(a,b)|\cos\theta$$

where θ is an angle between $\mathbf{u}$ and $\nabla f(a,b)$ (Figure 7). This gives the next theorem.

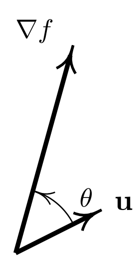


FIGURE 7

†The symbol ∇ is called "nabla" or "del."

Theorem 2 *Suppose that $\nabla f(a,b)$ is not the zero vector. Then* **(a)** *the maximum directional derivative of f at (a,b) is $|\nabla f(a,b)|$ and occurs for $\mathbf{u}$ with the same direction as $\nabla f(a,b)$,* **(b)** *the minimum directional derivative of f at (a,b) is $-|\nabla f(a,b)|$ and occurs for $\mathbf{u}$ with the opposite direction as $\nabla f(a,b)$, and* **(c)** *the directional derivative of f at (a,b) is zero for $\mathbf{u}$ with either of the two directions perpendicular to $\nabla f(a,b)$.*

Part (c) of this theorem implies that $\nabla f(a,b)$ is perpendicular to the level curve of f through the point (a,b).

Example 6 **(a)** What is the maximum directional derivative of $g(x,y) = y^2e^{2x}$ at $(2,-1)$ and in the direction of what unit vector does it occur? **(b)** What is the minimum directional derivative of g at $(2,-1)$ and in the direction of what unit vector does it occur?

Answer: (a) The maximum directional derivative is $\sqrt{8}\,e^4$ and occurs in the direction of $\mathbf{u} = \dfrac{\langle 1,-1\rangle}{\sqrt{2}}$.

(b) The minimum directional derivative is $= -\sqrt{8}\,e^4$ and occurs in the direction of $\mathbf{u} = \dfrac{\langle -1,1\rangle}{\sqrt{2}}$.

Example 7 Give the two unit vectors $\mathbf{u}$ such that the function $z = g(x,y)$ of Example 6 has zero derivatives at $(2,-1)$ in the direction of $\mathbf{u}$.

Answer: The directional derivative is zero in the directions of $\mathbf{u} = \dfrac{\langle -1,-1\rangle}{\sqrt{2}}$ and $\mathbf{u} = \dfrac{\langle 1,1\rangle}{\sqrt{2}}$.

Example 8 **(a)** Draw the gradient vector of $f(x,y) = xy$ at $(1,2)$ and the level curve of f through that point. **(b)** Draw $\nabla f(-3,1)$ and the level curve of f through $(-3,1)$.

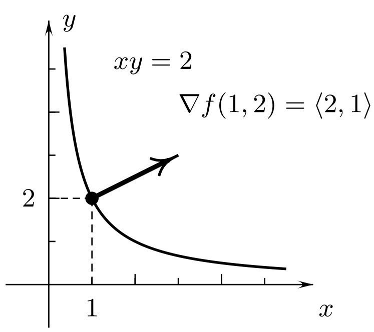


Figure A8a

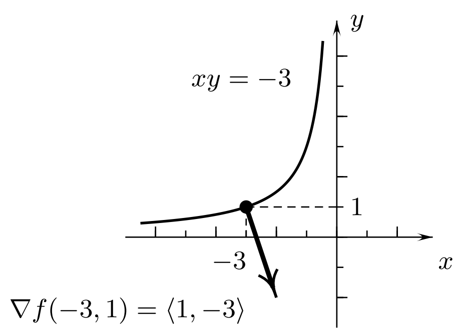


Figure A8b

Interactive Examples

Work the following Interactive Examples on Shenk's web page, http//www.math.ucsd.edu/˜ashenk/:[†]

Section 14.5: Examples 1 through 6

[†]The chapter and section numbers on Shenk's web site refer to his calculus manuscript and not to the chapters and sections of the textbook for the course.