Math 20C. Lecture Examples.

Section 14.5, Part 2. Directional derivatives and gradient vectors in space[†]

The definitions and results in Part 1 of this section concerning gradient vectors and directional derivatives with two variables can be converted to the three-variable case by allowing for the extra variable in the formulas.

Definition 1 **(a)** *The gradient vector of $w = f(x,y,z)$ at (a,b,c) is*

$$\nabla f(a,b,c) = \langle f_x(a,b,c), f_y(a,b,c), f_z(a,b,c)\rangle . \tag{1}$$

(b) *The directional derivative $D_u f(a,b,c)$ of $z = f(x,y,z)$ at (a,b,c) in the direction of the unit vector $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ is the t-derivative of the cross section $w = f(a+tu_1, b+tu_2, b+tu_3)$ at $t = 0$.*

The directional derivative $D_u f(a,b,c)$ is the rate of change of f with respect to distance in the direction of the vector $\mathbf{u}$.

Theorem 1 **(a)** *The directional derivative $D_u f(a,b,c)$ of $w = f(x,y,z)$ is equal to the dot product of the gradient of f at (a,b,c) with the unit vector $\mathbf{u}$:*

$$\begin{aligned} D_u f(a,b,c) &= \nabla f(a,b,c)\cdot \mathbf{u} \\ &= f_x(a,b,c)u_1 + f_y(a,b,c)u_2 + f_z(a,b,c)u_3. \end{aligned} \tag{2}$$

(b) *If the gradient vector $\nabla f(a,b,c)$ is not zero, then the greatest directional derivative of f at (a,b,c) is $|\nabla f(a,b,c)|$ and is in the direction of $\nabla f(a,b,c)$; the least directional derivative at (a,b,c) is $-|\nabla f(a,b,c)|$ and is in the direction opposite $\nabla f(a,b,c)$; and the directional derivatives at (a,b,c) in the directions perpendicular to $\nabla f(a,b,c)$ are zero.*

Example 1 **(a)** What is the gradient of $f(x,y,z) = xyz$ at $(1,2,3)$? **(b)** What is the directional derivative of f at $(1,2,3)$ in the direction toward $(2,3,4)$? **(c)** What is the greatest directional derivative of f at $(1,2,3)$?

Answer: (a) $\nabla f(1,2,3) = \langle 6,3,2\rangle$ **(b)** The directional derivative of f at $(1,2,3)$ in the direction toward $(2,3,4)$ is $\dfrac{18}{\sqrt{14}}$. **(c)** The greatest directional derivative of f at $(1,2,3)$ is 7.

[†]Lecture notes to accompany Section 14.5, Part 2 of *Calculus, Early Transcendentals* by Rogawski.

Gradient vectors and level surfaces

Theorem 2 *If $\nabla f(a,b,c)$ is not the zero vector, then it is perpendicular to the level surface of f through (a,b,c) and to the tangent plane at (a,b,c) to that level surface (Figure 1). The tangent plane has the normal vector $\nabla f(a,b,c)$ and the equation*

$$f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c) = 0. \tag{3}$$

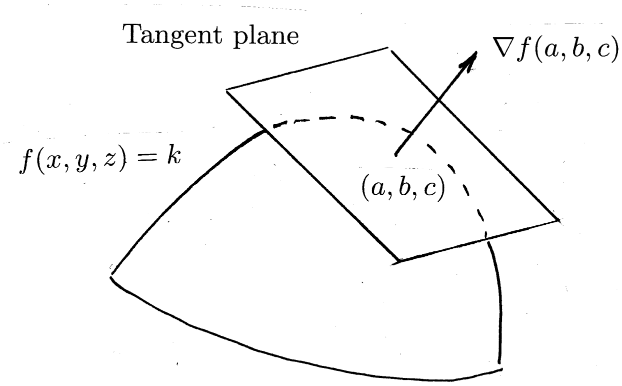

FIGURE 1

Example 2 Give an equation of the tangent plane at the point $(2,2,3)$ on the hyperboloid of two sheets $x^2+y^2-z^2=-1$

Answer: Tangent plane: $4(x-2)+4(y-2)-6(z-3)=0$ or $2x+2y-3z=-1$

Interactive Examples

Work the following Interactive Examples on Shenk's web page, http//www.math.ucsd.edu/˜ashenk/:†

Section 14.7: Examples 4, 5, and 6

†The chapter and section numbers on Shenk's web site refer to his calculus manuscript and not to the chapters and sections of the textbook for the course.