Math 20C. Lecture Examples

Section 14.7, Part 1. Maxima and minima: The first-derivative test

Example 1 **The function $\mathbf{g(x,y) = 2x^2 + y^2 - 2xy - 2y}$ has a global minimum. Find it.**

SOLUTION:

Because $z = g(x,y)$ is defined for all (x,y), the global minimum is a local minimum. •

$g_x = \frac{\partial}{\partial x}(2x^2 + y^2 - 2xy - 2y) = 4x - 2y = 2(2x - y)$ • $g_y = \frac{\partial}{\partial y}(2x^2 + y^2 - 2xy - 2y) = 2y - 2x - 2$ •

$g_x(x,y) = 2(2x - y)$ is zero if $y = 2x$ and then $g_y(x,y) = 2y - 2x - 2 = 2(2x) - 2x - 2 = 2x - 2$ is zero if $x = 1$.

• The one critical point is $(1,2)$ and the global minimum is $g(1,2) = 2(1^2) + 2^2 - 2(1)(2) - 2(2) = -2$.

Example 2 **Find, without using calculus, the global maximum of $\mathbf{f(x,y,z) = \frac{1}{(x-y)^2 + 1} + e^{-z^2}}$ and where it occurs.**

SOLUTION:

$g(x,y) = \frac{1}{(x-y)^2 + 1}$ has the global maximum of 1 along the line $y = x$ • $h(z) = e^{-z^2}$ has the global maximum of 1 at $z = 0$ • The global maximum of f is 2 at the points $(x,x,0)$ for all x.

Example 3 **The function $\mathbf{M(x,y) = \frac{-5y}{x^2 + y^2 + 1}}$, whose graph is shown in Figure 1, has a global maximum and a global minimum. What are they and where do they occur?**

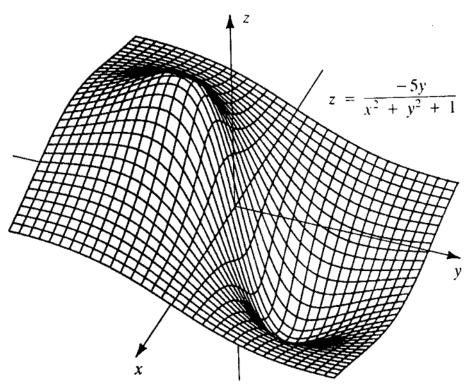

FIGURE 1

SOLUTION:

$$\frac{\partial M}{\partial x} = \frac{\partial}{\partial x}\left[\frac{-5y}{x^2 + y^2 + 1}\right] = \frac{\partial}{\partial x}[-5y(x^2 + y^2 + 1)^{-1}] = 5y(x^2 + y^2 + 1)^{-2}\frac{\partial}{\partial x}(x^2 + y^2 + 1)$$

$$= \frac{10xy}{(x^2 + y^2 + 1)^2} \bullet$$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left[\frac{-5y}{x^2 + y^2 + 1}\right] = \frac{(x^2 + y^2 + 1)\frac{\partial}{\partial y}(-5y) - (-5y)\frac{\partial}{\partial y}(x^2 + y^2 + 1)}{(x^2 + y^2 + 1)^2}$$

$$= \frac{-5(x^2 + y^2 + 1) + 5y(2y)}{(x^2 + y^2 + 1)^2} = \frac{5(y^2 - x^2 - 1)}{(x^2 + y^2 + 1)^2} \bullet$$

The critical points are the solutions (x,y) of $\begin{cases} xy = 0 \\ y^2 - x^2 - 1 = 0. \end{cases}$ •

For the first equation to be satisfied, either x or y must be zero. •
For $y = 0$, the second equation becomes $-x^2 - 1 = 0$, which has no solutions •
For $x = 0$, the second equation reads $y^2 - 1 = 0$ and has the solutions $y = \pm 1$. •
Critical points: $(0, 1)$ and $(0, -1)$ • Because $z = M(x, y)$ is defined for all (x, y), its global maximum and minimum must be local maximum and minimum and occur at the critical points. •
$M(0,1) = \dfrac{-5(1)}{1^2 + 0^2 + 1} = -\frac{5}{2}$ • $M(0,-1) = \dfrac{-5(-1)}{1^2 + 0^2 + 1} = \frac{5}{2}$ •
The global maximum is $\frac{5}{2}$ at $(0, -1)$ and the global minimum is $-\frac{5}{2}$ at $(0, 1)$.

Example 3 **Suppose that rectangular boxes with no tops (Figure 2) are to be manufactured so that each has a volume of 6 cubic feet. The boxes are to be made from material that costs 6 dollars per square foot for the bottoms, 2 dollars per square foot for the fronts and backs, and 1 dollar per square foot for the sides. What dimensions would minimize the cost of manufacturing each box? (You can assume that the minimum exists.)**

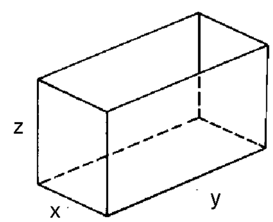

FIGURE 2

SOLUTION:

Denote the width of the front and back of the box by x, the length of the sides by y, and the height by z, all measured in feet, as in Figure 2 • The area of the bottom of the box is xy, the combined area of the front and back is $2xz$, and the combined area of the sides is $2yz$. •
[Cost of the box] = [Cost of the bottom] + [Cost of the front and back] + [Cost of the sides]

$$= [\,xy \text{ square feet}]\left[6\,\frac{\text{dollars}}{\text{square foot}}\right] + [\,2xz \text{ square feet}]\left[2\,\frac{\text{dollars}}{\text{square foot}}\right]$$

$$+[\,2yz \text{ square feet}]\left[1\,\frac{\text{dollar}}{\text{square foot}}\right] = 6xy + 4xz + 2yz \text{ dollars} \quad \bullet$$

Because the volume xyz of the box is to be 6 cubic feet, $xyz = 6$ and $z = \dfrac{6}{xy}$ •

$$C(x,y) = 6xy + 4xz + 2yz = 6xy + 4x\left(\frac{6}{xy}\right) + 2y\left(\frac{6}{xy}\right) = 6xy + \frac{24}{y} + \frac{12}{x} \quad \bullet$$

$$C_x = \frac{\partial}{\partial x}[6xy + 24y^{-1} + 12x^{-1}] = 6y - 12x^{-2} = 6(y - 2x^{-2}) \quad \bullet$$

$$C_y = \frac{\partial}{\partial y}[6xy + 24y^{-1} + 12x^{-1}] = 6x - 24y^{-2} = 6(x - 4y^{-2}) \quad \bullet$$

At a critical point, $\begin{cases} y - 2/x^2 & = 0 \\ x - 4/y^2 & = 0 \end{cases}$ • The first equation gives $y = 2/x^2$, which, when substituted into the second equation, yields $x = \dfrac{4}{(2/x^2)^2}$ • $x = x^4$ • $x(1 - x^3) = 0$ •
x cannot be 0. • $x = 1$ and $y = 2$ • The only critical point of $C(x, y)$ with $x > 0, y > 0$ is $(1, 2)$, • The cost is a minimum for $x = 1, y = 2$, for which $z = 6/(xy) = 3$. • The boxes should be manufactured to be 1 foot wide, 2 feet long, and 3 feet high.

Example 5 **Find the four points on the surface $\mathbf{z = 1/(xy)}$ that are closest to the origin $\mathbf{(0,0,0)}$ in xyz-space by finding the minimum of the square of the distance from $\mathbf{(x, y, (xy)^{-1})}$ on the surface to the origin. How far are they from the origin?**

SOLUTION:

Set $g(x,y) = [\text{Distance}]^2 = x^2 + y^2 + x^{-2}y^{-2}$ • $\begin{cases} g_x = 2x - 2x^{-3}y^{-2} = 2x(1 - x^{-4}y^{-2}) \\ g_y = 2y - 2x^{-2}y^{-3} = 2y(1 - x^{-2}y^{-4}) \end{cases}$ •

$w = g(x,y)$ is not defined if $x = 0$ or $y = 0$. • Solve $\begin{cases} x^{-4}y^{-2} = 1 \\ x^{-2}y^{-4} = 1. \end{cases}$ • $\begin{cases} x^4y^2 = 1 \\ x^2y^4 = 1 \end{cases}$ •

Multiply the first equation by y^2 and the second by x^2. • $x^2 = y^2$ • $y = \pm x$ • Either equation gives $x^6 = 1$ • $x = \pm 1$ • Critical points: $(\pm 1, \pm 1)$ • $g(\pm 1, \pm 1) = 1^2 + 1^2 + (1^2)(1^{-2}) = 3$ • $z(1,1) = 1, z(-1,1) = -1, z(1,-1) = -1, z(-1,-1) = 1$ • The closest points to the origin on the surface are $(1,1,1), (1,-1,-1), (-1,1,-1)$, and $(-1,-1,1)$ at a distance $\sqrt{3}$ from the origin.

Example 6 **What are the values of $\mathbf{x}$ and $\mathbf{y}$ such that the sum of the squares of the distances from the point $\mathbf{(x, y)}$ to the four points $\mathbf{(a_1, b_1), (a_2, b_2), (a_3, b_3)}$, and $\mathbf{(a_4, b_4)}$ is a minimum?**

Answer: $x = \frac{1}{4}(a_1 + a_2 + a_3 + a_4)$ • $y = \frac{1}{4}(b_1 + b_2 + b_3 + b_4)$ (The averages)

SOLUTION:

The sum of the square of the distances is

$S(x,y) = (x - a_1)^2 + (y - b_1)^2 + (x - a_2)^2 + (y - b_2)^2 + (x - a_3)^2 + (y - b_3)^2 + (x - a_4)^2 + (y - b_4)^2$ •

$S_x(x,y) = 2(x - a_1) + 2(x - a_2) + 2(x - a_3) + 2(x - a_4)$ •

$S_y(x,y) = 2(y - b_1) + 2(y - b_2) + 2(y - b_3) + 2(y - b_4)$ •

At the minimum $S_x = 0$ and $S_y = 0$ •

$x = \frac{1}{4}(a_1 + a_2 + a_3 + a_4)$ • $y = \frac{1}{4}(b_1 + b_2 + b_3 + b_4)$ (The averages of the coordinates of the four points)

Interactive Examples

Work the following Interactive Examples on Shenk's web page, http//www.math.ucsd.edu/˜ashenk/:[‡]

Section 15.1: Examples 1–6

[‡]The chapter and section numbers on Shenk's web site refer to his calculus manuscript and not to the chapters and sections of the textbook for the course.