(8/15/11)

Math 20C. Lecture Examples

Section 14.7, Part 2. Maxima and minima: The Second-Derivative Test

The second-degree Taylor polynomial approximation of $y = f(x)$ at $x = x_0$ is

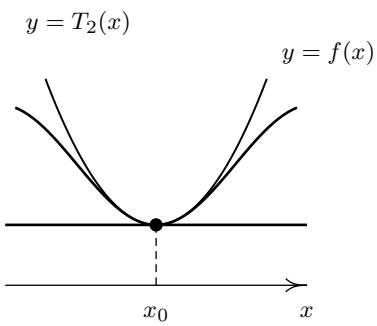
$$T_2(x) = f(x_0) + f'(x_0)(x - x_0) + \tfrac{1}{2}f''(x_0)(x - x_0)^2.$$

This polynomial has the same value and the same first and second derivatives as $y = f(x)$ at x_0 and, consequently, is the second-degree polynomial that best approximates $y = f(x)$ near x_0.

If x_0 is a critical point of $y = f(x)$, then $f'(x_0) = 0$ and the Taylor polynomial is

$$T_2(x) = f(x_0) + \tfrac{1}{2}f''(x_0)(x - x_0)^2.$$

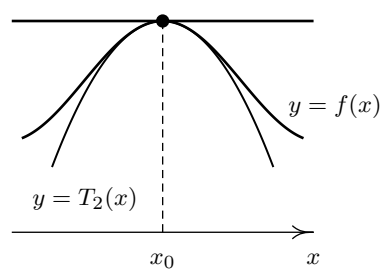
If $f''(x_0)$ is positive, then the graph of T_2 is a parabola that opens upward, as in Figure 1, and f has a local minimum at x_0. If $f''(x_0)$ is negative, then the graph of T_2 is a parabola that opens downward, as in Figure 4, and f has a local maximum at x_0.



$y = T_2(x)$ opens upward

Local minimum

FIGURE 1



$y = T_2(x)$ opens downward

Local maximum

FIGURE 2

To study local maxima and minima in the case of two variables, we approximate functions $z = f(x, y)$ by their second-degree Taylor polynomials.

Definition 1 *The second-degree Taylor approximation of $z = f(x, y)$ at $P_0 = (x_0, y_0)$ is*

$$\begin{aligned} T_2(x, y) = {} & f(P_0) + f_x(P_0)(x - x_0) + f_y(P_0)(y - x_0) \\ & + \tfrac{1}{2}f_{xx}(P_0)(x - x_0)^2 + f_{xy}(P_0)(x - x_0)(y - y_0) + \tfrac{1}{2}f_{yy}(P_0)(y - y_0)^2. \end{aligned} \tag{1}$$

The Taylor polynomial **(1)** has the same value and the same first- and second-order derivatives as f at $P_0 = (x_0, y_0)$ and is the second-degree polynomial that best approximates f near that point.

Example 1 **Give the second-degree Taylor polynomial approximation of $\mathbf{f(x,y) = 1 - \cos x \cos y}$ at $\mathbf{x = 0, y = 0}$.**

SOLUTION:

$f_x = \dfrac{\partial}{\partial x}(1 - \cos x \cos y) = \sin x \cos y$ • $f_y = \dfrac{\partial}{\partial y}(1 - \cos x \cos y) = \cos x \sin y$ •

$f_{xx} = \dfrac{\partial}{\partial x}(\sin x \cos y) = \cos x \cos y$ • $f_{xy} = \dfrac{\partial}{\partial y}(\sin x \cos y) = -\sin x \sin y$ •

$f_{yy} = \dfrac{\partial}{\partial y}(\cos x \sin y) = \cos x \cos y$ •

$f(0,0) = 1 - \cos(0)\cos(0) = 0$ •

$f_x(0,0) = \sin(0)\cos(0) = 0$ • $f_y(0,0) = \cos(0)\sin(0) = 0$ •

$f_{xx}(0,0) = \cos(0)\cos(0) = 1$ • $f_{xy}(0,0) = -\sin(0)\sin(0) = 0$ • $f_{yy}(0,0) = \cos(0)\cos(0) = 1$ •

$T_2(x,y) = f(0,0) + f_x(0,0)x + f_y(0,0)y + \frac{1}{2}f_{xx}(0,0)x^2 + f_{xy}(0,0)xy + \frac{1}{2}f_{yy}(0,0)y^2$ •

$T_2(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$

The graph of $f(x,y) = 1 - \cos x \cos y$ from Example 1 is in Figure 3. The graph of its Taylor polynomial approximation is the circular paraboloid $z = T_2(x,y)$ shown in Figure 4. The function f has a local minimum at $x = 0, y = 0$ because the Taylor polynomial has a global minimum there.

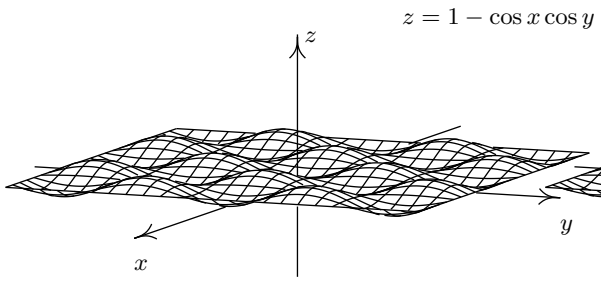


FIGURE 3

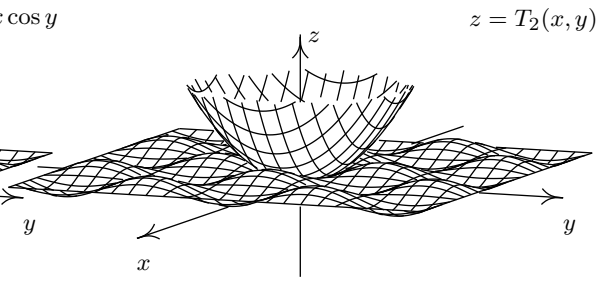
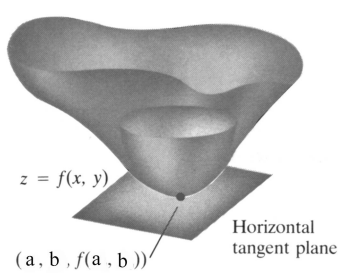


FIGURE 4

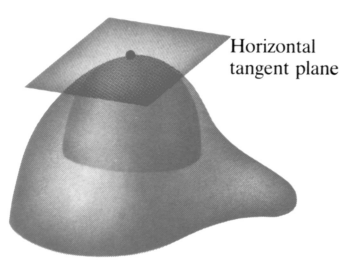
This approach can be applied to any function. Suppose that (x_0, y_0) is a critical point of f. Then $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$ and the Taylor polynomial approximation of f at (x_0, y_0) is

$$T_2(x,y) = f(x_0,y_0) + \tfrac{1}{2}f_{xx}(x_0,y_0)(x-x_0)^2 + f_{xy}(x_0,y_0)(x-x_0)(y-y_0) + \tfrac{1}{2}f_{yy}(x_0,y_0)(y-y_0)^2. \qquad \textbf{(2)}$$

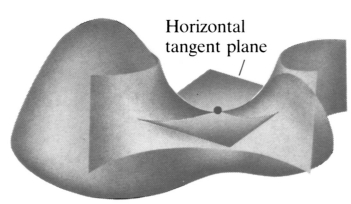
If the graph of T_2 is an elliptic paraboloid that opens upward as in Figure 5, then f has a local minimum at $x = x_0, y = y_0$; if the graph of T_2 is an elliptic paraboloid that opens downward as in Figure 6, then f has a local maximum at $x = x_0, y = y_0$; and if the graph of T_2 is a hyperbolic paraboloid as in Figure 7, then f has a SADDLE POINT, which is neither a local maximum nor local minimum, at $x = x_0, y = y_0$. These geometric ideas are the basis of the Second-Derivative Test with two variables.



Local minimum
FIGURE 5



Local maximum
FIGURE 6



Saddle point
FIGURE 7

Example 2 **Figures 8 and 9 show the graph of $\mathbf{f = -x^4 - y^4 - 4xy + \frac{1}{16}}$ and its level curves. Use the Second-Derivative Test to classify its critical points.**

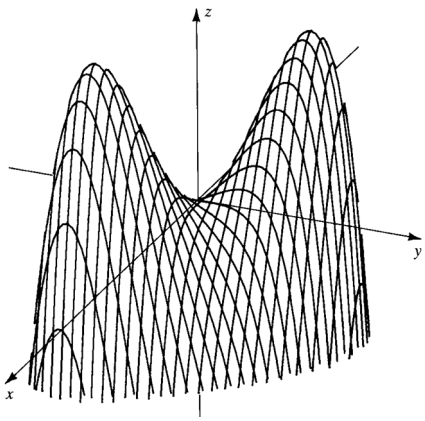


FIGURE 8

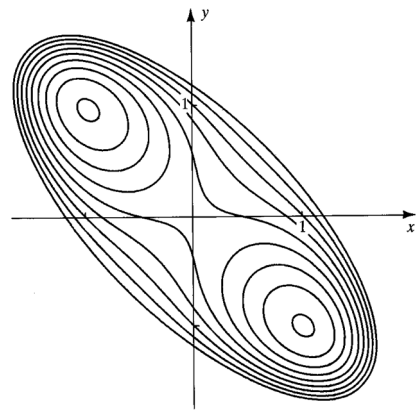


FIGURE 9

SOLUTION:

$f_x = \dfrac{\partial}{\partial x}(-x^4 - y^4 - 4xy + \frac{1}{16}) = -4x^3 - 4y = -4(x^3 + y)$ •

$f_y = \dfrac{\partial}{\partial y}(-x^4 - y^4 - 4xy + \frac{1}{16}) = -4y^3 - 4x = -4(y^3 + x)$ •

Critical points: $\begin{cases} x^3 + y = 0 \\ y^3 + x = 0 \end{cases}$ • $y = -x^3, x = -y^3$ • Substitute the first equation into the second: $x = -(-x^3)^3$ • $x - x^9 = 0$ • $x(1 - x^8) = 0$ • $x = 0, 1, -1$ • Since $y = -x^3$, the critical points are $(0,0), (1,-1)$, and $(-1,1)$. •

$f_{xx} = \dfrac{\partial}{\partial x}(-4x^3 - 4y) = -12x^2$ • $f_{xy} = \dfrac{\partial}{\partial y}(-4x^3 - 4y) = -4$ • $f_{yy} = \dfrac{\partial}{\partial y}(-4y^3 - 4x) = -12y^2$ •

See the table below. • The function f has a saddle point at $(0,0)$ because the discriminant $AC - B^2$ is negative there. • It has local maxima at $(1,-1)$ and at $(-1,1)$ because $AC - B^2$ is positive and A and C are negative at those points.

Critical point	$A = f_{xx} = -12x^2$	$B = f_{xy} = -4$	$C = f_{yy} = -12y^2$	$AC - B^2$
$(0,0)$	0	-4	0	$(0)(0) - (-4)^2 = -16$
$(1,-1)$	-12	-4	-12	$-12(-12) - 4^2 = 128$
$(-1,1)$	-12	-4	-12	$-12(-12) - 4^2 = 128$

Example 3 **Figures 10 and 11 show the graph of $\mathbf{f = -2x^3 - 3y^4 + 6xy^2}$ and its level curves. Use the Second-Derivative Test to classify its critical points.**

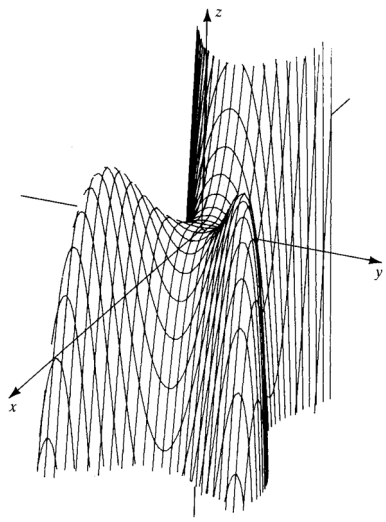


FIGURE 10

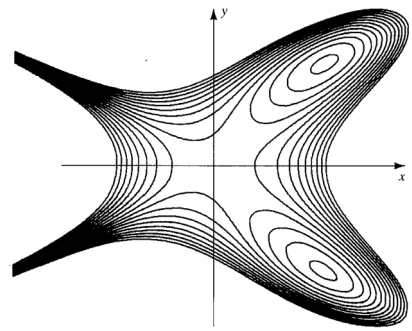


FIGURE 11

SOLUTION:

$f = -2x^3 - 3y^4 + 6xy^2$ • $f_x = -6x^2 + 6y^2 = -6(x^2 - y^2)$ • $f_y = -12y^3 + 12xy = -12y(y^2 - x)$ •
Critical points: $\begin{cases} x^2 - y^2 = 0 \\ y(y^2 - x) = 0. \end{cases}$ • The second equation gives $y = 0$ or $x = y^2$.
For $y = 0$, the first equation gives $x = 0$ and the critical point $(0, 0)$. •
For $x = y^2$, the first equation is $y^4 - y^2 = 0$ or $y^2(y^2 - 1) = 0$ and gives $y = 0, y = 1, y = -1$ with $x = y^2$.
• Critical points: $(0, 0), (1, 1), (1, -1)$ • $f_{xx} = -12x$ • $f_{xy} = 12y$ • $f_{yy} = -36y^2 + 12x$ •
See the table below.

Critical point	$A = f_{xx}$ $= -12x$	$B = f_{xy}$ $= 12y$	$C = f_{yy}$ $= -36y^2 + 12x$	$AC - B^2$	Type
$(0, 0)$	0	0	0	0	The test fails
$(1, 1)$	-12	12	-24	144	Local maximum
$(1, -1)$	-12	-12	-24	144	Local maximum

Interactive Examples

Work the following Interactive Examples on Shenk's web page, http//www.math.ucsd.edu/˜ashenk/:[‡]

Section 15.2: Examples 1–3

[‡]The chapter and section numbers on Shenk's web site refer to his calculus manuscript and not to the chapters and sections of the textbook for the course.