## Math 20C. Lecture Examples

## Section 14.7, Part 2

Imagine that the curve in Figure 1 is a mirror and that a viewer at point $F_{2}$ is looking at the image in the mirror of an object at point $F_{1}$. According to FERMAT'S PRINCIPLE from physics, the image will be the point $P$ on the mirror such that the total distance

$$
\begin{equation*}
f(P)=\overline{P F_{1}}+\overline{P F_{2}} \tag{1}
\end{equation*}
$$

that the light travels from the object to a point $P$ on the mirror to the viewer is a minimum. For each number $c$ that is greater than the distance $\overline{F_{1} F_{2}}$ between the object and the viewer, the level curve $\overline{P F_{1}}+\overline{P F_{2}}=c$ of the distance $f(P)$ is an ellipse. Figure 2 shows eight of these ellipses. (The ellipse $\overline{P F_{1}}+\overline{P F_{2}}=c$ could be drawn by fastening the ends of a string of length $c$ at $F_{1}$ and $F_{2}$ and running a pencil point around inside the taut string.)


The value of $f(P)$ is greater on larger ellipses, so the minimum value of $f(P)$ for $P$ on the mirror can be determined by finding the smallest ellipse that touches the mirror. $P_{0}$ is the point where that ellipse touches the mirror. A slightly larger ellipse would intersect the mirror at two nearby points, and a line through these points would be a secant line to the mirror and to the ellipse. As that ellipse shrinks down to the smallest ellipse that touches the mirror, the secant line becomes the tangent line to the mirror and to the ellipse at $P$, so both curves have the same tangent line at that point and are tangent to each other there.

To express this principle with formulas, we introduce $x y$-axes, as in Figure 3, and let $f(x, y)$ be the sum (1) of the distances from $P=(x, y)$ to $F_{1}$ and $F_{2}$. We assume that the mirror $C$ is a level curve $g(x, y)=c$ of another function with a nonzero gradient vector.

## FIGURE 3



At the point $P_{0}=\left(x_{0}, y_{0}\right)$ where the smallest ellipse and $C$ are tangent, $\nabla f\left(x_{0}, y_{0}\right)$ is perpendicular to the ellipse, which is the level curve of $f$, and $\nabla g\left(x_{0}, y_{0}\right)$ is perpendicular to $C$, which is the level curve of $g$. Since the curves are tangent to each other at $\left(x_{0}, y_{0}\right)$, the two gradient vectors are parallel, and there is a number $\lambda$ such that $\nabla f\left(x_{0}, y_{0}\right)=\lambda \nabla g\left(x_{0}, y_{0}\right)$. The number $\lambda$ is called a LaGRange multiplier.

This illustrates the following result for finding maximima and minima of a general function $z=f(x, y)$ on a curve.

Theorem 1 Suppose that $C$ is a level curve of $z=g(x, y)$, that $\nabla g(x, y)$ is not zero on $C$, and that $z=g(x, y)$ and $z=f(x, y)$ have continuous first derivatives near $C$. Then, if $z=f(x, y)$ has a local maximum or a local minimum for $(x, y)$ on $C$, then that maximum or minimum occurs at a point $\left(x_{0}, y_{0}\right)$ where for some number $\lambda$,

$$
\left\{\begin{align*}
\nabla f\left(x_{0}, y_{0}\right) & =\lambda \nabla g\left(x_{0}, y_{0}\right)  \tag{2}\\
g\left(x_{0}, y_{0}\right) & =c
\end{align*}\right.
$$

The curve $C$ in Theorem 1 is called the constraint curve. There are three scalar equations in (2) since the vector equation has two components. The three unknowns are $x_{0}, y_{0}$, and $\lambda$.

The number $\lambda$ in (2) is zero if $\nabla f\left(x_{0}, y_{0}\right)$ is the zero vector and $\left(x_{0}, y_{0}\right)$ is a critical point of $f$.
Example 1 Use Lagrange multipliers to find the maximum and minimum values of $f(x, y)=x-2 y+1$ on the ellipse $x^{2}+3 y^{2}=21$ and the points where they occur.
Solution Define $g(x, y)=x^{2}+3 y^{2}$ so that the ellipse is the level curve $g=21$. - By Theorem 1, $\nabla f=\lambda \nabla g$ for some constant $\lambda$ at any point where $f$ is a maximum or a minimum. • $\nabla f=\left\langle f_{x}, f_{y}\right\rangle=\langle 1,-2\rangle \bullet \nabla g=\left\langle g_{x}, g_{y}\right\rangle=\langle 2 x, 6 y\rangle \bullet\langle 1,-2\rangle=\lambda\langle 2 x, 6 y\rangle \bullet$
$\left\{\begin{array}{cc}1 & =2 \lambda x \\ -2 & =6 \lambda y\end{array} \bullet x=\frac{1}{2 \lambda}, y=\frac{-1}{3 \lambda} \bullet\right.$ Since $x^{2}+3 y^{2}=21,\left(\frac{1}{2 \lambda}\right)^{2}+3\left(\frac{-1}{3 \lambda}\right)^{2}=21 \bullet$ $\frac{\frac{1}{4}+\frac{1}{3}}{\lambda^{2}}=21 \bullet \frac{7}{12} \lambda^{-2}=21 \bullet \lambda^{2}=\frac{1}{36} \bullet \lambda= \pm \frac{1}{6} \bullet$
Setting $\lambda=\frac{1}{6}$ in the formulas $x=\frac{1}{2 \lambda}$ and $y=\frac{-1}{3 \lambda}$ gives $x=3$ and $y=-2$.
Setting $\lambda=-\frac{1}{6}$ gives $x=-3$ and $y=2$. - The maximum and minimum of $f$ on the ellipse occur at $(3,-2)$ or $(-3,2)$. Since $f(3,-2)=3-2(-2)+1=8$ and $f(-3,2)=-3-2(2)+1=-6$, the maximum of $f$ on the ellipse is 8 at $(3,-2)$ and the minimum is -6 at $(-3,2)$.

Example 2 Sketch the ellipse in Example 1 and the level curves of $f$ where the maximum and minimum occur.
Solution The ellipse $x^{2}+3 y^{2}=21$ intersects the $x$-axis at $x= \pm \sqrt{21} \doteq \pm 4.58$ and intersects the $y$-axis at $y= \pm \sqrt{7} \doteq \pm 2.65$ (Figure 4). - The function $f=x-2 y+1$ has the value 8 on the line $x-2 y+1=8$ (whose equation can be written $y=\frac{1}{2} x-\frac{7}{2}$ ) and has the value -6 on $x-2 y+1=-6$ (whose equation can be written $y=\frac{1}{2} x+\frac{7}{2}$ ). - These lines are are tangent to the ellipse at $(-3,2)$ and $(3,-2)$.

FIGURE 4


Example 3 Use Lagrange multipliers to find the rectangle of perimeter 12 that has the largest area (Figure 5).

FIGURE 5


Solution Let $x$ denote the width and $y$ the height of the rectangle, as in Figure 5. - Its area is $A(x, y)=x y$ and its perimeter is $P(x, y)=2 x+2 y$. • We want the maximum of $A(x, y)$ subject to the constraint $P(x, y)=12$.

$$
\begin{aligned}
& \nabla A=\left\langle\frac{\partial}{\partial x}(x y), \frac{\partial}{\partial y}(x y)\right\rangle=\langle y, x\rangle \\
& \nabla P=\left\langle\frac{\partial}{\partial x}(2 x+2 y), \frac{\partial}{\partial y}(2 x+2 y)\right\rangle=\langle 2,2\rangle
\end{aligned}
$$

The maximum of $A$ on $P=12$ occurs where $\langle y, x\rangle=\lambda\langle 2,2\rangle$ with some number $\lambda$. $\bullet$ $y=2 \lambda, \quad x=2 \lambda \bullet$ Substitute these formulas in the constraint equation: $2(2 \lambda)+2(2 \lambda)=12$ - $8 \lambda=12$ - $\lambda=\frac{3}{2}$ - $x=2\left(\frac{3}{2}\right)=3, y=2\left(\frac{3}{2}\right)=3$ - The rectangle with perimeter 12 and maximum area is a square with width 3 and height 3 .

In this case, the constraint curve is the line $2 x+2 y=12$ and the level curves of $A=x y$ are hyperbolas (Figure 6).

FIGURE 6


Example 4 Use Lagrange multipliers to find the dimensions of the rectangle of area 9 that has the shortest perimeter.
Solution Here we want the minimum of $P(x, y)=2 x+2 y$ for $A(x, y)=9 . \bullet \nabla P=\langle 2,2\rangle$ and $\nabla A=\langle y, x\rangle$ - $\langle 2,2\rangle=\lambda\langle y, x\rangle \bullet 2=\lambda y, 2=\lambda x \bullet x=2 / \lambda, y=1 / \lambda$ - Since $x y=9,4 / \lambda^{2}=9 \bullet \lambda= \pm \frac{2}{3}$

- Since $x$ and $y$ are positive, $\lambda=\frac{2}{3} \bullet x=\frac{2}{\frac{2}{3}}=3 \bullet y=\frac{2}{\frac{2}{3}}=3$ - The rectangle of area 9 and minimum perimeter is the square of width 3 and height 3 .

In this case the constraint curve is the hyperbola $x y=9$ and the level curves are the lines $2 x+2 y=c$ shown in Figure 7.

FIGURE 7


Example 5
Solution

Example 6

Solve Example 3 by finding the minimum of a function of one variable.
The constraint condition $2 x+2 y=12$ gives $y=6-x$. - Substituting this equation in the formula $A=x y$ for the area gives $A(x)=x(6-x)=6 x-x^{2}$ as a function of $x$ alone. $\bullet$ $A^{\prime}(x)=6-2 x=2(3-x)$ is zero at $x=3$, positive for $x<3$, and negative for $x>3$. - The area is a maximum for $x=3$, for which $y=6-x=3$. The rectangle of perimeter 12 with the maximum area is a square of width 3 and height 3, as we saw in Example 3.
Figure 8 shows level curves of the yield of corn, measured in thousand of pounds per acre, that a farmer will obtain if he applies $x$ acre-feet of irrigation water and $y$ pounds of fertilizer per acre during the growing season. (An acre foot of water would cover an acre one-foot deep.) Suppose that the water costs $\$ 60$ per acre-foot, the fertilizer costs 9 dollars per pound, and the farmer has $\$ 180$ to invest per acre for water and fertilizer. Approximately how much water and how much fertilizer should he buy to maximize the yield of corn?


FIGURE 8


FIGURE 9

Solution It would cost $C(x, y)=60 x+9 y$ dollars for $x$ acre-feet of water and $y$-pounds of fertilizer, so that $x$ and $y$ must satisfy $x \geq 0, y \geq 0,60 x+9 y \leq 180$ and consequently $(x, y)$ must be in the shaded triangle of Figure 9. As the drawing shows, the maximum yield occurs on the boundary line $60 x+9 y=180$. To estimate where it occurs, we draw a plausible level curve of the yield tangent to the line, as in Figure 9, and use the coordinates of the point of tangency. The farmer should use approximately 1.75 acre-feet of water and 8.3 pounds of fertilizer for each acre.

## Lagrange multipliers with three variables (not required)

Theorem 2 Suppose that $\Sigma$ is a level surface of $g(x, y, z)$, that $\nabla g(x, y, z)$ is not zero on $\Sigma$, and that $z=f(x, y, z)$ and $z=g(x, y, z)$ have continuous first derivatives in an open set containing $\Sigma$. If $f$, restricted to $\Sigma$, has a local maximum or local minimum on $\Sigma$, then that local maximum or minimum occurs at a point ( $x_{0}, y_{0}, z_{0}$ ) where, for some number $\lambda$

$$
\begin{equation*}
\nabla f\left(x_{0}, y_{0}, z_{0}\right)=\lambda \nabla g\left(x_{0}, y_{0}, z_{0}\right) \tag{3}
\end{equation*}
$$

The Lagrange multiplier $\lambda$ is zero in (3) if $\left(x_{0}, y_{0}, z_{0}\right)$ is a critical point of $f$.
Example $7 \quad$ Find the maximum and minimum values of $f=6 x+3 y+2 z-5$ on the ellipsoid $4 x^{2}+2 y^{2}+z^{2}=70$.
Solution Set $g=4 x^{2}+2 y^{2}+z^{2}$, so the ellipsoid is the level surface $g=70$. Find the gradient vectors,

$$
\begin{aligned}
& \nabla f=\left\langle\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle(6 x+3 y+2 z-5)=\langle 6,3,2\rangle \\
& \nabla g=\left\langle\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle\left(4 x^{2}+2 y^{2}+z^{2}\right)=\langle 8 x, 4 y, 2 z\rangle
\end{aligned}
$$

The condition $\nabla f=\lambda \nabla g$ reads $\langle 6,3,2\rangle=\lambda\langle 8 x, 4 y, 2 z\rangle$ and gives the three scalar equations,

$$
6=8 \lambda x, 3=4 \lambda y, 2=2 \lambda z
$$

These yield

$$
\begin{equation*}
x=\frac{3}{4 \lambda}, y=\frac{3}{4 \lambda}, z=\frac{1}{\lambda} . \tag{4}
\end{equation*}
$$

so that the condition $4 x^{2}+2 y^{2}+z^{2}=70$ gives

$$
4\left(\frac{3}{4 \lambda}\right)^{2}+2\left(\frac{3}{4 \lambda}\right)^{2}+\left(\frac{1}{\lambda}\right)^{2}=70
$$

This equation simplifies to to $\frac{70}{16} \lambda^{-2}=70$ and then yields $\lambda^{2}=\frac{1}{16}$, so that $\lambda= \pm \frac{1}{4}$.
With $\lambda=\frac{1}{4}$, equations (4) give $x=3, y=3, z=4$, and with $\lambda=-\frac{1}{4}$ they give $x=-3, y=-3, z=-4$.

The maximum and minimum of $f$ on $\Sigma$ must occur at $(3,3,4)$ or $(-3,-3,-4)$. Calculate

$$
\begin{aligned}
f(3,3,4) & =6(3)+3(3)+2(4)-5=30 \\
f(-3,-3,-4) & =6(-3)+3(-3)+2(-4)-5=-40
\end{aligned}
$$

The maximum is 30 and the minimum is -40 .

## Interactive Examples

Work the following Interactive Examples on Shenk's web page, http//www.math.ucsd.edu/ ashenk/. (The chapter and section numbers on this web site do not to the chapters and sections of the textbook for the course.)

Section 15.3: Examples 1, 2

