Math 20C. Lecture Examples (Revised)

Sections 15.1 and 15.2. Double integrals†

A region R in the xy-plane is BOUNDED if it can be enclosed in a sufficiently large circle. Its boundary is PIECEWISE SMOOTH if it consists of a finite number of graphs $y = y(x)$ or $x = x(y)$ of functions that are defined on finite closed intervals and have continuous first derivatives on those intervals.

A PARTITION,

$$R_1, R_2, \ldots, R_N \tag{1}$$

of such a region is created by slicing it with piecewise smooth curves. Figure 2 shows a partition of the region R of Figure 1 into four subregions. To measure the size of a subregion we use its DIAMETER, which is the diameter of the smallest circle that contains it.

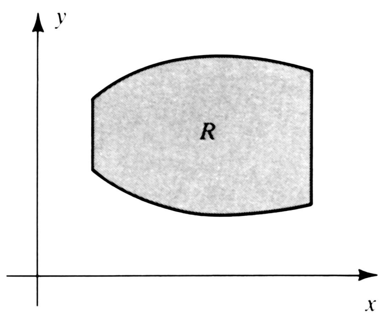

FIGURE 1

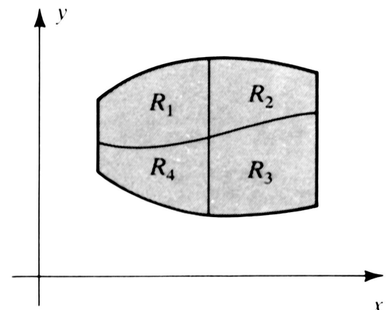

FIGURE 2

The INTERIOR of a region is obtained by removing any boundary points from it. The CLOSURE is obtained by adding any boundary points that are not in it. A function $z = f(x, y)$ is PIECEWISE CONTINUOUS on a region if there is a partition of the region such that in the interior of each subregion, f is equal to a function that is continuous on the closure of the subregion.

Definition 1 (Riemann sums) *Suppose that $z = f(x, y)$ is piecewise continuous on a bounded region R with a piecewise-smooth boundary and that* **(1)** *is a partition of R. A* RIEMANN SUM *for $\iint_R f(x, y)\, dx\, dy$ corresponding to the partition* **(1)** *is a sum of the form*

$$\sum_{j=1}^{N} f(x_j, y_j)\ [\text{Area of } R_j] \tag{2}$$

where for each $j = 1, 2, \ldots, N$, (x_j, y_j) is a point in R_j where f is defined.

DOUBLE INTEGRALS over such regions are the limits of Riemann sums:

Definition 2 (Double integrals) *If $z = f(x, y)$ is piecewise continuous on a bounded region R with a piecewise-smooth boundary, then the double integral $\iint_R f(x, y)\, dx\, dy$ is the limit of Riemann sums* **(2)** *as the number of subintervals in the partition tends to infinity and their diameters tend to zero.*

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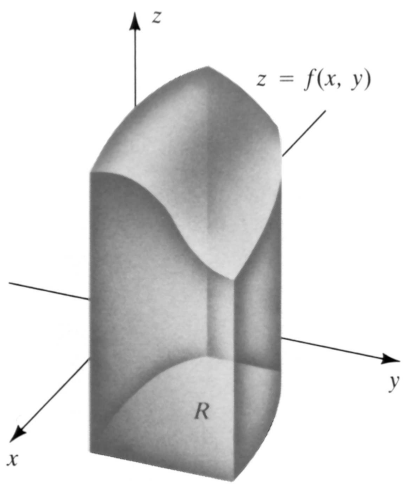
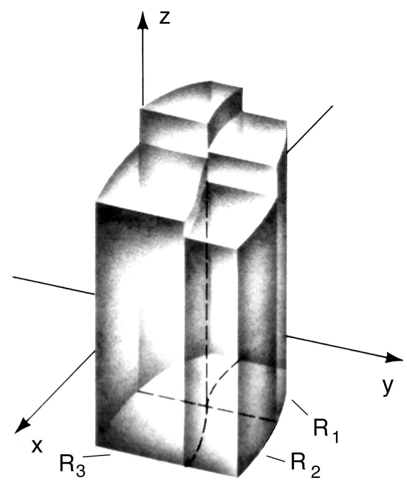
†Lecture notes to accompany Sections 15.1 and 15.2 of *Calculus, Early Transcendentals* by Rogawski

For the region R of Figure 1, the partition of Figure 2, and the positive function $z = f(x, y)$ of Figure 3, the Riemann sum is equal to the total volume of four prism-shaped solids as in Figure 4, whose sides are vertical, whose bases are the four subregions of the partition, and whose horizontal tops intersect the graph of f at $x = x_j, y = y_j$. Because such collections of solids approximate the solid under the graph, we are led to the following definition.



$$[\text{Volume}] = \iint_R f(x, y)\, dx\, dy$$

FIGURE 3



[Volume] = A Riemann sum]

FIGURE 4

Definition 3 (Volume) *If R is a bounded region in an xy-plane with a piecewise-smooth boundary and $z = f(x, y)$ is piecewise continuous and has nonnegative values on R, then the volume of the solid above R and below the graph $z = f(x, y)$ is $\iint_R f(x, y)\, dx\, dy$ (Figure 3).*

Definition 3 gives the same results as other definitions of volume in cases where the volumes are also defined by formulas from geometry or by using integrals with one variable.

Example 1 **What is the value of $\iint_{\mathbf{R}} \sqrt{\mathbf{9 - x^2 - y^2}}\, \mathbf{dx\, dy}$ where R is the disk $\mathbf{x^2 + y^2 \le 9}$?**

Answer: $\iint_R \sqrt{9 - x^2 - y^2}\, dx\, dy = [\text{Volume of the hemisphere of radius 3 in Figure A1}] = \frac{1}{2}(\frac{4}{3})\pi(3^3) = 18\pi$

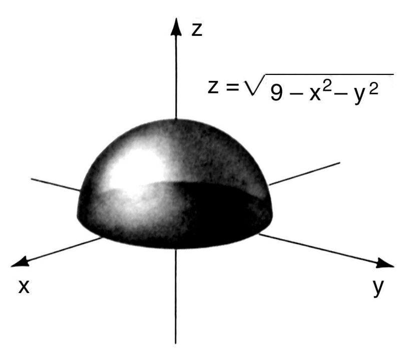


Figure A1

Theorem 1 (Iterated integrals) **(a)** *Suppose that R is the region in an xy-plane bounded on the bottom by the graph $y = g(x)$ and on the top by $y = h(x)$ for $a \le x \le b$ (Figure 5), and that g and h are piecewise continuous on $[a, b]$. Then for any function $z = f(x, y)$ that is piecewise continuous on R,*

$$\iint_R f(x,y)\,dx\,dy = \int_{x=a}^{x=b} \left\{ \int_{y=g(x)}^{y=h(x)} f(x,y)\,dy \right\} dx. \tag{3}$$

(b) *If R is bounded on the left by the curve $x = g(y)$ and on the top by $x = h(y)$ for $a \le y \le b$ (Figure 6), and if g and h are piecewise continuous on $[a, b]$, then for any function $z = f(x, y)$ that is piecewise continuous on R,*

$$\iint_R f(x,y)\,dx\,dy = \int_{y=a}^{y=b} \left\{ \int_{x=g(y)}^{x=h(y)} f(x,y)\,dx \right\} dy. \tag{4}$$

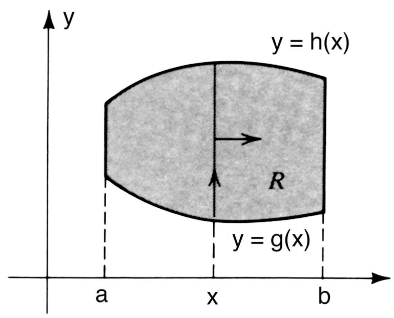


FIGURE 5

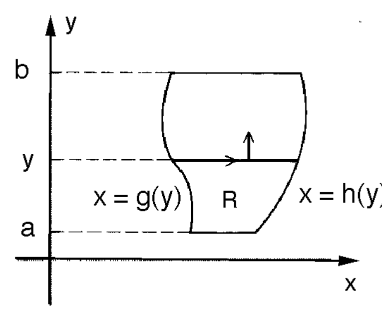
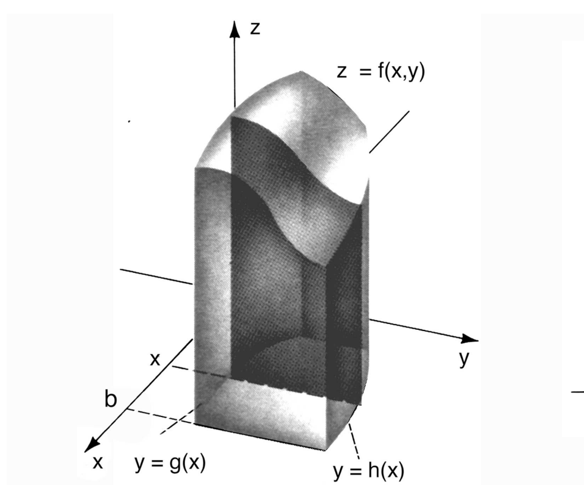


FIGURE 6

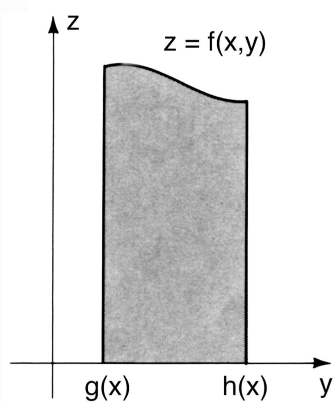
The integration procedures in **(3)** and **(4)** are indicated by the lines and arrows in Figures 5 and 6.

Formula **(1)** for a positive function $z = f(x, y)$ as in Figure 7 is a consequence of the procedure for finding volumes by slicing, since the inner integral in **(1)** is equal to the volume of the cross section of the solid in the plane perpendicular to the x-axis at x that is shown in Figure 8.



$$[\text{Volume}] = \int_{x=a}^{x=b} A(x)\,dx$$

FIGURE 7



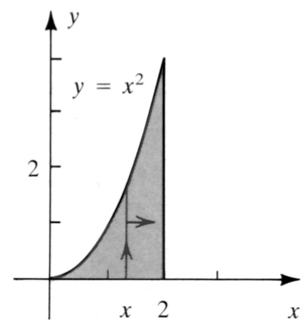
$$A(x) = \int_{y=g(x)}^{y=h(x)} f(x,y)\,dy$$

FIGURE 8

Example 2 **Evaluate $\iint_R 2xy\,dx\,dy$, where R is the region bounded by the curve $y = x^2$ and the lines $y = 0$ and $x = 2$ by performing a y-integration first.**

Answer: Figure A2 • $\iint_R 2xy\;dx\,dy = \frac{32}{3}$

Figure A2



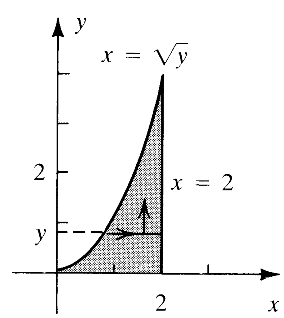
SOLUTION:

Figure A2 shows the region of integration with arrows to indicate the integration procedure. •
$$\iint_R 2xy\;dx\,dy = \int_{x=0}^{x=2}\left\{\int_{y=0}^{y=x^2} 2xy\;dy\right\}\,dx = \int_{x=0}^{x=2}\Big[xy^2\Big]_{y=0}^{y=x^2}\,dx$$
$$= \int_{x=0}^{x=2}\left\{[x(x^2)^2] - [x(0)^2]\right\}\,dx = \int_{x=0}^{x=2} x^5\;dx = \Big[\tfrac{1}{6}x^6\Big]_{x=0}^{x=2} = [\tfrac{1}{6}(2^6)] - [\tfrac{1}{6}(0^6)] = \tfrac{32}{3}$$

Example 3 **Evaluate the integral of Example 2 by performing an x-integration first.**

Answer: Figure A3 • $\iint_R 2xy\;dx\,dy = \frac{32}{3}$

Figure A3



SOLUTION:

Figure A3 • The left side of the region has the equation $x = \sqrt{y}$, the right side is $x = 2$, and the region extends from $y = 0$ to $y = 4$. • $\iint_R f(x,y)\;dx\,dy = \int_{y=0}^{y=4}\int_{x=\sqrt{y}}^{x=2} 2xy\;dx\,dy$
$$= \int_{y=0}^{y=4}\Big[x^2y\Big]_{x=\sqrt{y}}^{x=2}\,dy = \int_{y=0}^{y=4}\left\{[(2)^2y] - [(\sqrt{y})^2y]\right\}\,dy = \int_{y=0}^{y=4}(4y - y^2)\;dy$$
$$= \Big[2y^2 - \tfrac{1}{3}y^3\Big]_{y=0}^{y=4} = [2(4)^2 - \tfrac{1}{3}(4)^3] - [0] = 32 - \tfrac{64}{3} = \tfrac{32}{3}$$

Reversing the order of integration

In the next example, the inner integration in the given iterated integral cannot be done but the double integral can be evaluated by first reversing the order of integration.

Example 4 **Evaluate** $\displaystyle \int_0^8 \int_{x^{1/3}}^2 \sin(y^4)\, dy\, dx.$

Answer: Figures A4a and A4b • $\displaystyle \int_0^8 \int_{x^{1/3}}^2 \sin(y^4)\, dy\, dx = \tfrac{1}{4} - \tfrac{1}{4}\cos(16)$

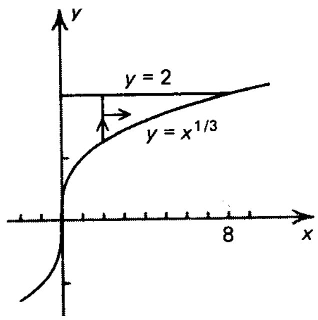


Figure A4a

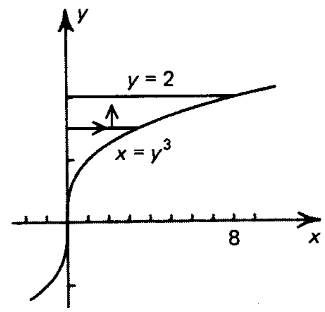


Figure A4b

SOLUTION:

$\displaystyle \int_0^8 \int_{x^{1/3}}^2 \sin(y^4)\, dy\, dx = \int_{x=0}^{x=8} \int_{y=x^{1/3}}^{y=2} \sin(y^4)\, dy\, dx$ • The region of integration is bounded by the curve $y = x^{1/3}$, the line $y = 2$ and the y-axis, as shown in Figure A4a. •

$y = x^{1/3} \iff x = y^3$ • Figure A4b • The given integral equals $\displaystyle \int_{y=0}^{y=2} \int_{x=0}^{x=y^3} \sin(y^4)\, dx\, dy$

$\displaystyle = \int_{y=0}^{y=2} \Big[x\sin(y^4) \Big]_{x=0}^{x=y^3} dy = \int_{y=0}^{y=2} y^3 \sin(y^4)\, dy$ •

$\displaystyle \int y^3 \sin(y^4)\, dy = \tfrac{1}{4} \int \sin(y^4)\, 4y^3\, dy = \tfrac{1}{4} \int \sin u\, du = -\tfrac{1}{4}\cos u + C = -\tfrac{1}{4}\cos(y^4) + C$ •

The given integral equals $\displaystyle \Big[-\tfrac{1}{4}\cos(y^4) \Big]_{y=0}^{y=2} = [-\tfrac{1}{4}\cos(2^4)] - [-\tfrac{1}{4}\cos(0^4)] = \tfrac{1}{4} - \tfrac{1}{4}\cos(16)$

Theorem 2 (Volumes of solids between graphs) *Suppose that $z = g(x,y)$ and $z = h(x,y)$ are piecewise continuous in a bounded region R with a piecewise-smooth boundary and that $g(x,y) \leq h(x,y)$ for all points (x,y) in R where $y = g(x,y)$ and $h(x,y)$ are defined. Let V be the solid consisting of the points above $z = g(x,y)$ and below $z = h(x,y)$ for (x,y) in R (Figure 9). Then*

$$[\text{Volume of } V] = \iint_R [h(x,y) - g(x,y)]\, dx\, dy \qquad \textbf{(5)}$$

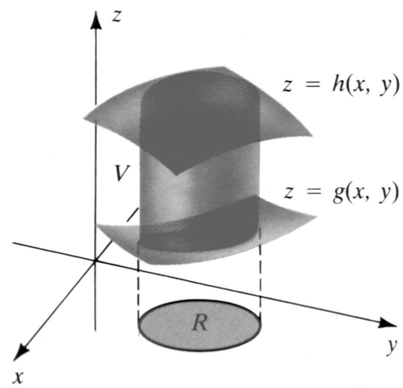


FIGURE 9

Example 5 **What is the volume of the solid V bounded by the surface $\mathbf{z = x^2 + y^2}$ and the plane $\mathbf{z = -2}$ for $\mathbf{0 \leq x \leq 2, 0 \leq y \leq x}$?**

Answer: [Volume] $= \frac{28}{3}$

SOLUTION:

The projection of V on the xy-plane is the triangle R: $0 \leq x \leq 1, 0 \leq y \leq x$ in Figure A5. • $-2 < x^2 + y^2$ for all (x,y). • [Volume] $= \iint_R [(x^2+y^2) - (-2)]\, dx\, dy$

$= \int_{x=0}^{x=2} \int_{y=0}^{y=x} (x^2 + y^2 + 2)\, dx\, dy = \int_{x=0}^{x=2} \left[x^2 y + \frac{1}{3}y^3 + 2y\right]_{y=0}^{y=x} = \int_{x=0}^{x=2} (x^3 + \frac{1}{3}x^3 + 2x)\, dx$

$= \int_{x=0}^{x=2} (\frac{4}{3}x^3 + 2x)\, dx = \left[\frac{1}{3}x^4 + x^2\right]_{x=0}^{x=2} = [\frac{1}{3}(2^4) + 2^2] - [0] = \frac{28}{3}$

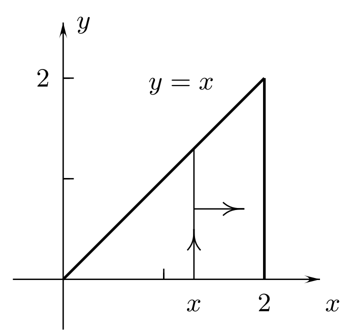


Figure A5

Definition 4 (Density, weight, and mass) *Suppose that a flat plate occupies the bounded region R with a piecewise-smooth boundary in an xy-plane and its density at (x, y), measured in weight or mass per unit area, is given by the piecewise-continuous function $\rho = \rho(x, y)$. Then the weight or mass of the plate is*

$$\iint_R \rho(x, y)\, dx\, dy. \tag{6}$$

Example 6 **A plate that occupies the region $\mathbf{R : 0 \le x \le 2, 0 \le y \le x^2}$ with distances measured in feet has density $\rho(\mathbf{x}, \mathbf{y}) = \mathbf{3xy^2}$ pounds per square foot at $(\mathbf{x}, \mathbf{y})$. How much does the plate weigh?**

Answer: [Weight] = 32 pounds

SOLUTION:

Figure A6 • $[\text{Weight}] = \iint_R 3xy^2\, dx\, dy = \int_{x=0}^{x=2} \int_{y=0}^{y=x^2} 3xy^2\, dy\, dx$

$= \int_{x=0}^{x=2} \left[xy^3 \right]_{y=0}^{y=x^2} dx = \int_{x=0}^{x=2} x(x^2)^3\, dx = \int_{x=0}^{x=2} x^7\, dx = \left[\frac{1}{8} x^8 \right]_{x=0}^{x=2}$

$= \frac{1}{8}(2^8) - \frac{1}{8}(0^8) = 2^5 = 32$ pounds

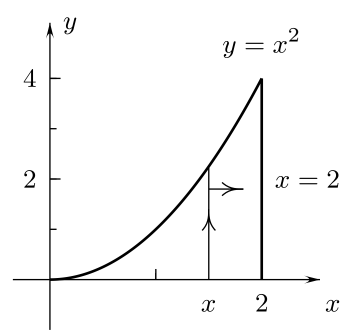
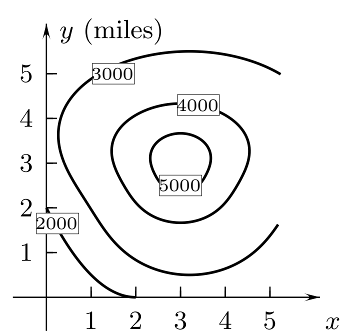


Figure A6

Example 7 **Figure 10 shows level curves of the population density $\mathbf{z = p(x, y)}$ (people per square mile) in a city. Find the approximate population in the region $\mathbf{R: 0 \le x \le 4, 0 \le y \le 4}$. Use a Riemann sum corresponding to a partition into four squares.**



Level curves of $z = p(x, y)$
(people per square mile)
FIGURE 10

Answer: One answer: The population is approximately 58,000.

SOLUTION:

[Population] $= \iint_R p(x,y)\,dy\,dx$ • One approach: Approximate the integral with the Riemann sum corresponding to the partition of R into the four equal square subregions in Figure A7 with the population density evaluated at their midpoints $(1,1), (3,1), (1,3)$, and $(3,3)$. • $p(1,1) \approx 2400$, $p(3,1) \approx 3400$, $p(1,3) \approx 3500$, $p(3,3) \approx 5200$ • The area of each subregion is $2(2) = 4$ square miles • [Riemann sum] $= p(1,1)(4) + p(1,3)(4) + p(3,1)(4) + p(3,3)(4)$ $\approx 2400(4) + 3400(4) + 3500(4) + 5200(4) = 58,000$ • The population is approximately 58,000.

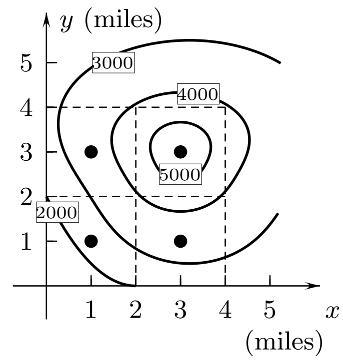


Figure A7

Definition 5 (Average value) *If $z = f(x,y)$ is piecewise continuous on the bounded region R with piecewise-smooth boundary, then*

$$[\text{The average value of } f \text{ on } R] = \frac{1}{[\text{Area of } R]} \iint_R f(x,y)\,dx\,dy.$$

Example 8 **What is the average value of $\mathbf{f(x,y) = ye^{y^2}}$ on the region R: $\mathbf{0 \le y \le \sqrt{x}, 0 \le x \le 1}$?**

Answer: [Average value] $= \frac{3}{4}(e-2)$

SOLUTION:

Figure A8 • $[\text{Area of } R] = \iint_R 1\,dx\,dy = \int_{x=0}^{x=1}\int_{y=0}^{y=\sqrt{x}} 1\,dy\,dx = \int_{x=0}^{x=1}\Big[\,y\,\Big]_{y=0}^{y=\sqrt{x}}\,dx$

$= \int_{x=0}^{x=1} x^{1/2}\,dx = \left[\tfrac{2}{3}x^{3/2}\right]_{x=0}^{x=1} = \tfrac{2}{3}$ •

$\iint_R ye^{y^2}\,dx\,dy = \int_{x=0}^{x=1}\int_{y=0}^{y=\sqrt{x}} ye^{y^2}\,dy\,dx = \int_{x=0}^{x=1}\left[\tfrac{1}{2}e^{y^2}\right]_{y=0}^{y=\sqrt{x}}\,dy\,dx = \int_{x=0}^{x=1}\tfrac{1}{2}(e^x-1)\,dx$

$= \left[\tfrac{1}{2}(e^x - x)\right]_{x=0}^{x=1} = \tfrac{1}{2}(e-1) - \tfrac{1}{2}(1-0) = \tfrac{1}{2}(e-2)$ • (This used $\int ye^{y^2}\,dy = \tfrac{1}{2}e^{y^2} + C$.)

$[\text{Average value}] = \dfrac{\tfrac{1}{2}(e-2)}{\tfrac{2}{3}} = \tfrac{3}{4}(e-2)$

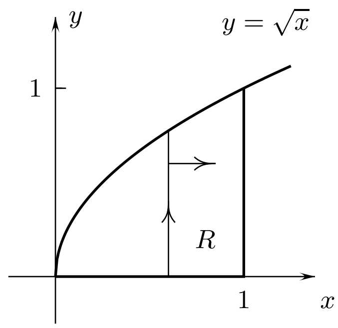


Figure A8

More practice

Example 9 **Evaluate $\iint_R 10x^4y\,dx\,dy$ with R the triangle with vertices $(0,0), (0,2)$, and $(1,1)$.**

Answer: $\iint_R 10x^4y\,dx\,dy = \int_{x=0}^{x=1}\int_{y=x}^{y=2-x} 10x^4y\,dy\,dx = \tfrac{2}{3}$

Example 10 **What is the value of $\iint_R e^x \sin y\,dx\,dy$ where R is the rectangle bounded by $x = -3, x = 4, y = 0$ and $y = 5$?**

Answer: $\iint_R e^x \sin y\,dx\,dy = \int_{x=-3}^{x=4}\int_{y=0}^{y=5} e^x \sin y\,dy\,dx = (1-\cos(5))(e^4 - e^{-3})$

Example 11 **Evaluate $\iint_R 3y^2\sqrt{x}\,dx\,dy$ with R bounded by $y = x^2, y = -x^2$, and $x = 4$.**

Answer: $\iint_R 3y^2\sqrt{x}\,dx\,dy = \int_{x=0}^{x=4}\int_{y=-x^2}^{y=x^2} 3y^2\sqrt{x}\,dy\,dx = \dfrac{1}{15}2^{17}$

Interactive Examples

Work the following Interactive Examples on Shenk's web page, http//www.math.ucsd.edu/˜ashenk/:‡

Section 16.1: Examples 3, 4, 5

Section 16.2: Examples 1, 2a, 3

‡The chapter and section numbers on Shenk's web site refer to his calculus manuscript and not to the chapters and sections of the textbook for the course.