## Math 20C. Lecture Examples.

## Section 15.3. Triple integrals ${ }^{\dagger}$

A solid $V$ in $x y z$-space is bOUNDED if it can be enclosed in a sphere. A surface is PIECEWISE SMOOTH if it consists of a finite number of graphs $z=g(x, y), y=g(x, z)$, or $x=g(y, z)$, where in each case $g$ is defined and has continuous first-order derivatives on a closed, bounded, plane region with a piecewise-smooth boundary.

A function $w=f(x, y, z)$ is piecewise continuous on $V$ if it is defined in all of $V$, except possibly at a finite number of points or on a finite number of piecewise-smooth curves or surfaces, and if there is a fixed partition $V_{1}, V_{2}, V_{3}, \ldots, V_{N}$ of $V$ such that in the interior of each $V_{j}, f(x, y, z)=F_{j}(x, y, z)$, where $F_{j}$ is continuous in the closure of $V_{j}$.

A Partition of a bounded solid $V$ with a piecewise-smooth boundary is obtained by using a finite number of piecewise-smooth surfaces to divide $V$ into subsolids,

$$
\begin{equation*}
V_{1}, V_{2}, V_{3}, \ldots, V_{N} \tag{1}
\end{equation*}
$$

The diameter of a subsolid is the diameter of the smallest sphere that contains it.
A Riemann sum for the triple integral $\iiint_{V} f(x, y, z) d x x y d z$ of the piecewise continuous function $f$ is formed from a partition (1) of $V$ by picking a point $\left(x_{j}, y_{j}, z_{j}\right)$ in $V_{j}$ for $j=1,2, \ldots, N$ such that $f\left(x_{j}, y_{j}, z_{j}\right)$ is defined. The Riemann sum is

$$
\begin{equation*}
\sum_{j=1}^{N} f\left(x_{j}, y_{j}, z_{j}\right)\left[\text { Volume of } V_{j}\right] \tag{2}
\end{equation*}
$$

The triple integral of $f$ over $V$ is a limit of such sums:
Definition 1 If $V$ is a bounded solid with a piecewise smooth boundary and $w=f(x, y, z)$ is piecewise continuous on $V$, then the TRIPLE INTEGRAL

$$
\begin{equation*}
\iiint_{V} f(x, y, z) d x d y d z \tag{3}
\end{equation*}
$$

is the limit of the Riemann sums (2) as the number of subsolids in the corresponding partitions (1) tends to infinity and their diameters tend to zero.
$\begin{array}{ll}\text { Example } 1 & \text { What is the geometric significance of } \iiint_{\mathbf{V}} \mathbf{1} \mathbf{d x} \mathbf{d y d z} \text { ? } \\ \text { Solution } & \iiint_{V} 1 d x d y d z \text { is equal to the volume of } V \text { since each each Riemann sum for the integral } \\ \text { equals the volume of } V .\end{array}$

[^0]
## Algebra of triple integrals

The next theorem lists properties of triple integrals that are similar to corresponding properties of single and double integrals. These formulas are consequences of the analogous formulas for the Riemann sums.

Theorem 1 (a) If $w=f(x, y, z)$ and $w=g(x, y, z)$ are piecewise continuous on a bounded solid $V$ with piecewisesmooth boundary, then for any constants $A$ and $B$,

$$
\begin{equation*}
\iiint_{V}[A f+B g] d x d y d z=A \iiint_{V} f d x d y d z+B \iiint_{V} g d x d y d z \tag{4}
\end{equation*}
$$

(b) If $w=f(x, y, z)$ is piecewise continuous in the union $V$ of bounded solids $V_{1}$ and $V_{2}$ with piecewise-smooth boundaries and nonintersecting interiors, then

$$
\begin{equation*}
\iiint_{V} f d x d y d z=\iiint_{V_{1}} f d x d y d z+\iiint_{V_{2}} f d x d y d z \tag{5}
\end{equation*}
$$

## Evaluating triple integrals by iteration

Suppose that $V$ is the solid in Figure 1 that is bounded on the top by the surface $z=h(x, y)$ and on the bottom by the surface $z=g(x, y)$ for $(x, y)$ in the bounded region $R$ with a piecewise-smooth boundary in the $x y$-plane. Integrals over $V$ can be found as iterated integrals with the following result:

FIGURE 1


Theorem 2 Suppose that $R$ is a bounded region with a piecewise-smooth boundary in the $x y$-plane and that $z=g(x, y)$ and $z=h(x, y)$ are piecewise continuous on $R$ and satisfy $g(x, y) \leq z \leq h(x, y)$ at all points in $R$ where they are defined. Suppose that $V$ is the solid between the graphs of $g$ and $h$ for $(x, y)$ in $R$. Then for any function $w=f(x, y, z)$ that is piecewise continuous on $V$,

$$
\begin{equation*}
\iiint_{V} f(x, y, z) d x d y d z=\iint_{R}\left\{\int_{z=g(x, y)}^{z=h(x, y)} f(x, y, z) d z\right\} d x d y \tag{6}
\end{equation*}
$$

In applying (6), we refer to $R$ as the projection of $V$ on the $x y$-plane.

Example 2 Evaluate $\iiint_{V} 3 x y d x d y d z$, where $V$ is the solid between the $x y$-plane and the hyperbolic paraboloid $z=x y$ for $0 \leq y \leq x, 0 \leq x \leq 1$.
Solution The projection of $V$ on the $x y$-plane is the triangle $R$ in Figure 2. - The base of $V$ is $z=0$ and its top is $z=x y$.

FIGURE 2

$$
\begin{aligned}
& \begin{array}{rl}
\iint_{V} 3 x y & d x
\end{array} d_{y} d z=\iint_{R} \int_{z=0}^{z=x y} 3 x y d z d y d x \\
&=\int_{x=0}^{x=1} \int_{y=0}^{y=x} \int_{z=0}^{z=x y} 3 x y d z d y d x \\
&=\int_{x=0}^{x=1} \int_{y=0}^{y=x}[3 x y z]_{z=0}^{z=x y} d y d x \\
&=\int_{x=0}^{x=1} \int_{y=0}^{y=x} 3 x^{2} y^{2} d y d x \\
&=\int_{x=0}^{x=1}\left[x^{2} y^{3}\right]_{y=0}^{y=x} d x \\
&= \int_{x=0}^{x=1} x^{5} d x=\left[\frac{1}{6} x^{6}\right]_{x=0}^{x=1}=\frac{1}{6}
\end{aligned}
$$

Example $3 \quad$ What is the value of $\iiint_{V}(x+y+z) d x d y d z$ if $V$ is the box defined by $0 \leq x \leq 3,0 \leq y \leq 2,0 \leq z \leq 1$ ?
Solution The description of the box $V$ shows that its top is formed by the plane $z=1$, its bottom is formed by the plane $z=0$, and its projection on the $x y$-plane is the rectangle $R: 0 \leq x \leq 3,0 \leq y \leq 2$ in Figure 3. Therefore,

FIGURE 3


$$
\begin{aligned}
\iiint_{V}(x+y & +z) d x d y d z=\iint_{R} \int_{z=0}^{z=1}(x+y+z) d z d y d x \\
= & \int_{x=0}^{x=3} \int_{y=0}^{y=2} \int_{z=0}^{z=1}(x+y+z) d z d y d x \\
= & \int_{x=0}^{x=3} \int_{y=0}^{y=2}\left[x z+y z+\frac{1}{2} z^{2}\right]_{z=0}^{z=1} d y d x \\
= & \int_{x=0}^{x=3} \int_{y=0}^{y=2}\left\{\left[x+y+\frac{1}{2}(1)^{2}\right]-[0]\right\} d y d x \\
= & \int_{x=0}^{x=3} \int_{y=0}^{y=2}\left(x+y+\frac{1}{2}\right) d y d x \\
= & \int_{x=0}^{x=3}\left[x y+\frac{1}{2} y^{2}+\frac{1}{2} y\right]_{y=0}^{y=2} d x \\
= & \int_{x=0}^{x=3}\left\{\left[2 x+\frac{1}{2}\left(2^{2}\right)+\frac{1}{2}(2)\right]-[0]\right\} d x=\int_{x=0}^{x=3}(2 x+3) d x \\
= & {\left[x^{2}+3 x\right]_{x=0}^{x=3}=\left[3^{2}+3(3)\right]-[0]=18 . \square }
\end{aligned}
$$

Definition 2 (Weight and mass) Suppose that $V$ is a bounded solid with a piecewise-smooth boundary that has density $\rho=\rho(x, y, z)$ at ( $x, y, z$ ), measured in weight (or mass) per unit volume, where $\rho$ is piecewise continuous on $V$. Then
$[$ Weight (or mass) of $V]=\iiint_{V} \rho(x, y, z) d x d y d z$.

Example $4 \quad$ The solid $V$ in $x y z$-space with distances measured in meters is bounded by $z=0, z=y, y=x^{2}$, and $y=1$. Its density at $(x, y, z)$ is $\rho(x, y, z)=8 y z$ kilograms per cubic meter. What is its mass?

Solution
[Mass] $=\iiint_{V} \rho(x, y, z) d x d y d z=\iiint_{V} 8 y z d x d y d z \bullet$ The projection of $V$ on the $x y$-plane is the region $R$ in Figure 4, which is bounded by the parabola $y=x^{2}$ and the line $y=1$ for $-1 \leq x \leq 1$. - The bottom of $V$ is $z=0$ (the $x y$-plane) and its top is the plane $z=y$.

FIGURE 4


$$
\begin{aligned}
{[\mathrm{Mass}] } & =\iiint_{V} 8 y z d x d y d z \\
& =\iint_{R} \int_{z=0}^{z=y} 8 y z d x d y d z \\
& =\int_{z=-1}^{x=1} \int_{y=x^{2}}^{y=1} \int_{z=0}^{z=y} 8 y z d z d y d x \\
& =\int_{z=1}^{x=1} \int_{y=x^{2}}^{y=1}\left[4 y z^{2}\right]_{z=0}^{z=y} d x d x \\
& =\int_{x=-1}^{x=1} \int_{y=x^{2}}^{y=1} 4 y^{3} d y d x \\
& =\int_{x=-1}^{x=1}\left[y^{4}\right]_{y=x^{2}}^{y=1} d x \\
& =\int_{x=-1}^{x=1}\left(1-x^{8}\right) d x \\
& =2 \int_{x=0}^{x=8}\left(1-x^{8}\right) d x \\
& =2\left[x-\frac{1}{9} x^{9}\right]_{x=0}^{x=1}=2\left(1-\frac{1}{9}\right) \\
& =\frac{16}{9} \operatorname{kilograms}
\end{aligned}
$$

Definition 3 For a piecewise-continuous function $f$ on a bounded solid $V$ with a piecewise-smooth boundary,
[Average value of $f$ in $V]=\frac{1}{[\text { Volume of } V]} \iiint_{V} f(x, y, z) d x d y d z$.

The next example has a special solution because it involves the integral of a product of functions of $x, y$, and $z$ over a box with sides parallel to the coordinate planes.
Example $5 \quad$ What is the average value of $f=x y^{3} z^{7}$ in the the box $V$ bounded by $x=0, x=2$,

$$
y=0, y=2, z=0 \text { and } z=2 ?
$$

Solution $\quad[$ Volume of $V]=2^{3}=8 \bullet$

$$
\begin{aligned}
& \iiint_{V} x y^{3} z^{7} d x d y d z=\int_{x=0}^{x=2} \int_{y=0}^{y=2} \int_{z=0}^{z=2} x y^{3} z^{7} d z d y d x \\
& =\int_{x=0}^{x=2} \int_{y=0}^{y=2} x y^{3} \int_{z=0}^{z=2} z^{7} d z d y d x \\
& =\int_{x=0}^{x=2} x \int_{y=0}^{y=2} y^{3} \int_{z=0}^{z=2} z^{7} d z d y d x \\
& =\left(\int_{z=0}^{z=2} z^{7} d z\right)\left(\int_{y=0}^{y=2} y^{3} d y\right)\left(\int_{x=0}^{x=2} x d x\right) \\
& =\left[\frac{1}{2} x^{2}\right]_{x=0}^{x=2}\left[\frac{1}{4} y^{4}\right]_{y=0}^{y=2}\left[\frac{1}{8} z^{8}\right]_{z=0}^{z=2} \\
& =(2)(4)(32)=256 \bullet[\text { Average value }]=\frac{256}{8}=32
\end{aligned}
$$

More practice
Example 6 Evaluate $\iiint_{V} z d x d y d z$ with $V$ bounded by the $x y$-plane, the cone $z=\sqrt{x^{2}+y^{2}}$, and the vertical planes $x= \pm 1, y= \pm 1$.

$$
\text { Answer: } \iiint_{V} z d x d y d z=\int_{x=-1}^{x=1} \int_{y=-1}^{y=1} \int_{z=0}^{z=\sqrt{x^{2}+y^{2}}} z d z d y d x=\frac{4}{3}
$$

Example $7 \quad$ What is the value of $\iiint_{V} 15 \sqrt{y z} d x d y d z$ if $V$ is the solid bounded by $z=0, z=y, y=$ $x^{2}, y=1$ ?
Answer: $\iiint_{V} 15 \sqrt{y z} d x d y d z=\int_{x=-1}^{x=1} \int_{y=x^{2}}^{y=1} \int_{z=0}^{z=y} 15 y^{1 / 2} z^{1 / 2} d z d y d x=\frac{40}{7}$.

## Interactive Examples

Work the following Interactive Examples on Shenk's web page, http//www.math.ucsd.edu/ ashenk/. (The chapter and section numbers on this web site do not to the chapters and sections of the textbook for the course.)

Section 16.4: Examples 1-4


[^0]:    ${ }^{\dagger}$ Lecture notes to accompany Section 15.3 of Calculus, Early Transcendentals by Rogawski

