

Math 20C. Lecture Examples.

Section 15.4. Integrals in polar coordinates[†]

Recall that the polar coordinates of any point $P = (x, y)$ other than the origin are $[r, \theta]$, where $r = \sqrt{x^2 + y^2}$ is the distance from P to the origin and θ is an angle from the positive x -axis to the line segment from the origin to the point (Figure 1) and the polar coordinates of the origin are $[0, \theta]$ for any angle θ . Also, rectangular coordinates can be calculated from polar coordinates with the formulas,

$$x = r \cos \theta, y = r \sin \theta.$$

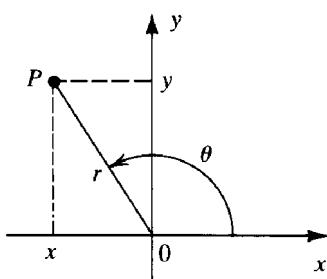


FIGURE 1

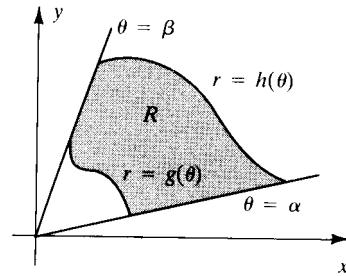


FIGURE 2

Theorem 1 Suppose that R is bounded by the curves $r = g(\theta)$ and $r = h(\theta)$ for $\alpha \leq \theta \leq \beta$ where $0 < \beta - \alpha \leq 2\pi$ and where $g(\theta)$ and $h(\theta)$ satisfy $0 \leq g(\theta) \leq h(\theta)$ and are piecewise continuous in the closed interval $[\alpha, \beta]$ (Figure 2). Then for any piecewise-continuous function $z = f(x, y)$ in R ,

$$\iint_R f(x, y) dx dy = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=g(\theta)}^{r=h(\theta)} f(r \cos \theta, r \sin \theta) r dr d\theta.$$

- Example 1**
- (a) Express $\iint_R y dx dy$ as an iterated integral in polar coordinates, where R is the region bounded by the x -axis and the half cardioid $r = 1 + \cos \theta, 0 \leq \theta \leq \pi$ in Figure 3.
 - (b) Evaluate the integral.

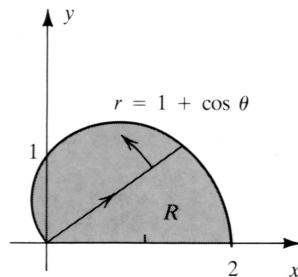


FIGURE 3

[†]Lecture notes to accompany Section 15.4 of *Calculus* by Rogawski

SOLUTION

$$\begin{aligned}
 \text{(a)} \quad & \iint_R y \, dx \, dy = \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=1+\cos\theta} (r \sin \theta) r \, dr \, d\theta \\
 \text{(b)} \quad & \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=1+\cos\theta} (r \sin \theta) r \, dr \, d\theta = \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=1+\cos\theta} r^2 \sin \theta \, dr \, d\theta \\
 & = \int_{\theta=0}^{\theta=\pi} \left[\frac{1}{3} r^3 \sin \theta \right]_{r=0}^{r=1+\cos\theta} d\theta = \int_{\theta=0}^{\theta=\pi} \frac{1}{3} (1 + \cos \theta)^3 \sin \theta \, d\theta \bullet \\
 u & = 1 + \cos \theta, du = -\sin \theta \, d\theta \bullet \quad \frac{1}{3} \int (1 + \cos \theta)^3 \sin \theta \, d\theta = -\frac{1}{3} \int (1 + \cos \theta)^3 (-\sin \theta \, d\theta) \\
 & = \frac{1}{3} \int u^3 \, du = -\frac{1}{12} u^3 + C = -\frac{1}{12} (1 + \cos \theta)^4 + C \bullet
 \end{aligned}$$

$$\iint_R y \, dx \, dy = \left[\frac{1}{12} (1 + \cos \theta)^4 \right]_0^{\pi} = -\frac{1}{12} (0^4 - 2^4) = \frac{4}{3}$$

Example 2Find the average value of $f(x, y) = \sqrt{x^2 + y^2 + 1}$ on the circle $R: x^2 + y^2 \leq 1$.

SOLUTION

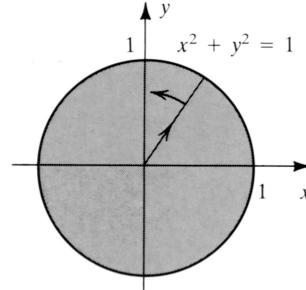
[Area of R] = $\pi(1)^2 = \pi$ • Figure 4

FIGURE 4

$$\begin{aligned}
 [\text{Average value}] &= \frac{1}{[\text{Area of } R]} \iint_R \sqrt{x^2 + y^2 + 1} \, dx \, dy \\
 &= \frac{1}{\pi} \iint_R \sqrt{x^2 + y^2 + 1} \, dx \, dy \\
 &= \frac{1}{\pi} \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \sqrt{r^2 + 1} r \, dr \, d\theta \\
 &= \frac{1}{\pi} \left(\int_{\theta=0}^{\theta=2\pi} d\theta \right) \int_{r=0}^{r=1} (1 + r^2)^{1/2} r \, dr \\
 &= \frac{2\pi}{\pi} \int_{r=0}^{r=1} (1 + r^2)^{1/2} r \, dr \\
 &= 2 \int_{r=0}^{r=1} (1 + r^2)^{1/2} r \, dr \bullet
 \end{aligned}$$

$$\begin{aligned}
 u &= 1 + r^2, du = 2r \, dr \bullet \quad 2 \int (1 + r^2)^{1/2} r \, dr = \int (1 + r^2)^{1/2} (2r \, dr) \\
 &= \int u^{1/2} \, du = \frac{2}{3} u^{3/2} + C = \frac{2}{3} (1 + r^2)^{3/2} + C \bullet
 \end{aligned}$$

$$\begin{aligned}
 [\text{Average value}] &= \left[\frac{2}{3} (1 + r^2)^{3/2} \right]_{r=0}^{r=1} \\
 &= \frac{2}{3} (2^{3/2} - 1)
 \end{aligned}$$

Example 3 Find the volume of the solid between the xy -plane and the hyperbolic paraboloid $z = xy$ for (x, y) in the region $R: 0 \leq y \leq \sqrt{x - x^2}$. The curve $y = \sqrt{x - x^2}$ is the semicircle in Figure 5 with the polar equation $r = \cos \theta$, $0 \leq \theta \leq \frac{1}{2}\pi$.

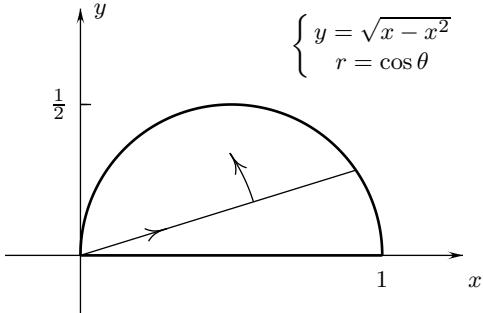


FIGURE 5

SOLUTION

Since $xy \geq 0$ for (x, y) in R ,

$$\begin{aligned}
 [\text{Volume}] &= \iint_R xy \, dx \, dy = \int_{\theta=0}^{\theta=\pi/2} \int_{r=0}^{r=\cos\theta} (r \cos \theta)(r \sin \theta) r \, dr \, d\theta \\
 &= \int_{\theta=0}^{\theta=\pi/2} \int_{r=0}^{r=\cos\theta} r^3 \cos \theta \sin \theta \, dr \, d\theta \\
 &= \int_{\theta=0}^{\theta=\pi/2} \left[\frac{1}{4} r^4 \cos \theta \sin \theta \right]_{r=0}^{r=\cos\theta} \, d\theta \\
 &= \frac{1}{4} \int_{\theta=0}^{\theta=\pi/2} \cos^5 \theta \sin \theta \, d\theta \\
 &= \frac{1}{4} \left[-\frac{1}{6} \cos^6 \theta \right]_{\theta=0}^{\theta=\pi/2} \quad (\text{using } u = \cos \theta, du = -\sin \theta \, d\theta) \\
 &= -\frac{1}{4} \left(0 - \frac{1}{6} \right) = \frac{1}{24}
 \end{aligned}$$

More practice

Example 4 Use polar coordinates to evaluate $\iint_R (x^2 + y^2)^{-2} \, dx \, dy$ where R is defined by the inequalities $2 \leq x^2 + y^2 \leq 4$.

$$\text{Answer: } \iint_R (x^2 + y^2)^{-2} \, dx \, dy = \int_{\theta=0}^{\theta=2\pi} \int_{r=\sqrt{2}}^{r=2} r^{-3} \, dr \, d\theta = \frac{1}{4}\pi$$

Example 5 What is the value of $\int_{-4}^4 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} e^{-x^2-y^2} \, dy \, dx$?

$$\begin{aligned}
 \text{Answer: } \int_{-4}^4 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} e^{-x^2-y^2} \, dy \, dx &= \iint_R e^{-x^2-y^2} \, dx \, dy = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=4} e^{-r^2} r \, dr \\
 &= \pi(1 - e^{-16}) \quad (\text{Here } R \text{ is the disk of radius 4 centered at the origin.})
 \end{aligned}$$

Interactive Examples

Work the following Interactive Examples on Shenk's web page, <http://www.math.ucsd.edu/~ashenk/>. (The chapter and section numbers on this web site do not correspond to the chapters and sections of the textbook for the course.)

Section 16.3: Examples 1–4