Chapter 0. Mathematical Models: Functions and Graphs (5/17/07)

The story of calculus goes back thousands of years. Mathematicians of the ancient world, including Pythagoras (c. 580 BC), Euclid (c. 300 BC), Archimedes (c. 287–212 BC), and Apollonius (c. 262–190 BC), developed a great deal of the mathematics that is used in calculus. This includes the theory of proportions, most of what is now known as plane geometry, the theory of conic sections, and results concerning tangent lines and areas of curved regions. They did all this without using letters for known and unknown quantities or equations relating quantities as in modern algebra, without representing points by coordinates or describing curves and surfaces with equations as in analytic geometry, and without using the modern concept of function.†

Modern calculus, in contrast, relies heavily on algebra and analytic geometry and on properties of power, exponential, logarithmic, and trigonometric functions. These topics are reviewed in this precalculus chapter. Read it carefully, answer the questions in the discussions, study the examples, and be sure you can work all of the tune-up exercises and any problems in the regular problem sets that you are assigned. You might also want to work through the problem tutorials on the web page for this text.‡

Section 0.1 deals with mathematical models, functions, and graphs. Notation and terminology for intervals of numbers are described and rules for solving inequalities are discussed in Section 0.2. The definitions, basic properties, and graphs of power functions, exponential functions, logarithms, and trigonometric functions are reviewed in Sections 0.3 through 0.5. Section 0.6 deals with sums, products, quotients, and compositions of these basic functions. Work through Sections 0.1 through 0.3 before or while you study Chapter 1, and work through Sections 0.4 through 0.6 before or while you study Chapter 2.

Section 0.1

Measuring and modeling: variables and functions

OVERVIEW: *To solve a problem using calculus, the first step is to set up a mathematical model in which the quantities that vary are represented by variables and functions. The procedures of calculus are applied, using arithmetic and algebra as required. Then the solution is interpreted in the context of the application. The notation and terminology concerning functions that are used in this process are discussed in this section and illustrated with applications to a variety of fields.*

Topics:

- ***Variables, functions, and graphs***
- ***Functions given by formulas, tables, and graphs***
- ***Finding formulas for mathematical models***
- ***When is a graph the graph of a function?***
- ***Change, percent change, and relative change***

Variables, functions, and graphs

Suppose we want to find the area of the 1978 silver dollar in Figure 1 and we know that its radius is 1.9 centimeters. We use the formula

$$A = \pi r^2 \ \text{ for } r \geq 0 \tag{1}$$

for the area of a circle of radius r (Figure 2), and we conclude that the area of the coin is $A = \pi(1.9)^2 = 3.61\pi$ square centimeters.

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†Highlights in the history of algebra and analytic geometry are given in the historical notes at the end of Section 0.6.

‡See http//www.math.ucsd.edu/˜ashenk/.

When we apply formula **(1)** for the area of a circle to the coin, we are using a MATHEMATICAL MODEL. Coins are, in fact, not perfectly circular—at least on a microscopic level—but we carry out our calculations with the area formula, under the assumption that its results will be sufficiently accurate for our purposes.

FIGURE 1

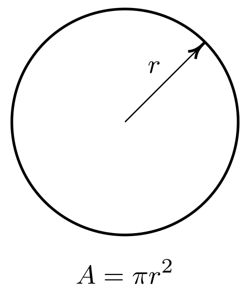


FIGURE 2

Such mathematical models use several special terms. Formula **(1)** defines the area as a FUNCTION of the radius. The VARIABLE of the function is r, and its DOMAIN is the set $r \geq 0$ of values of the variable for which the function is defined. The VALUE of the function is A and its RANGE is the set $A \geq 0$ of values of the function for r in its domain. The variable r of the function is also referred to as the INDEPENDENT VARIABLE, and the letter A that is used for values of the function is called the DEPENDENT VARIABLE.

The GRAPH of the function **(1)** in the rA-plane of Figure 3 consists of the points (r, A) with r in the domain and $A = \pi r^2$. Its shape can be predicted by the values of the function in the following table.† The function **(1)** is not defined for $r < 0$ and its graph in Figure 3 does not extend to the left of the vertical axis because circles do not have negative radii. We have chosen to include circles with zero radius in our mathematical model. Accordingly, a dot has been placed at the origin to indicate that the point at $r = 0$ is included in the graph.

r	0	1	2	3
$A = \pi r^2$	0	$\pi \doteq 3.14$	$4\pi \doteq 12.57$	$9\pi \doteq 20.27$

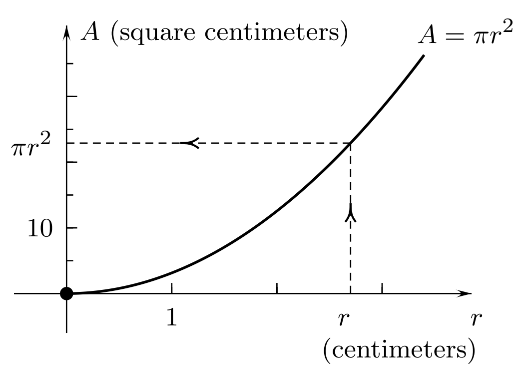


FIGURE 3

†The symbol $\doteq$ is used in this table and throughout this book for approximations that come from round-off errors with decimals. The symbol $\approx$ is used for other types of approximations.

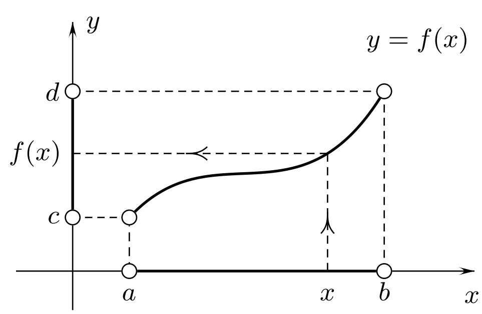
Figure 3 illustrates how values of functions are found from their graphs. Starting with a value of the variable r on the horizontal r-axis, we move vertically to the graph and then horizontally to the value $A = \pi r^2$ of the function, which is a number on the A-axis.

We can denote the area function of Figure 3 as the function "A" by writing $A(r) = \pi r^2$ for the area of the circle of radius r. This is an example of the following general definition.

Definition 1 *A* FUNCTION *f of one variable x is a rule that assigns a single number $f(x)$ to each x in a set D of numbers. D is the* DOMAIN *of the function, and $f(x)$ is the* VALUE OF THE FUNCTION *at x. The* RANGE *of the function is the set of its values. The* GRAPH *of the function in an xy-plane is the curve $y = f(x)$, which is the set of points with coordinates $(x, f(x))$ for x in the domain.*

This definition is illustrated by the function f of Figure 4. Its domain is the interval $a < x < b$ on the x-axis and its range is the interval $c < y < d$ on the y-axis. The open circles indicate that the function is not defined at the endpoints $x = a$ and $x = b$ of the interval and, consequently, the endpoints of the curve are not in the graph. Notice that the value $f(x)$ at x can be determined by finding x on the x-axis, moving vertically to the graph, and then horizontally to $f(x)$ on the y-axis. The graph is labeled $y = f(x)$ because it consists of all points (x, y) with x in the domain of the function and $y = f(x)$.

FIGURE 4



Question 1 How would we need to change our sketch of the graph of the area function in Figure 3 if in our mathematical model we considered only circles with positive radii?[†]

The next examples and question use the area function **(1)**.

Example 1 Is the area of a circle increased more by increasing its radius from 1 to 5 feet or by increasing the radius from 12 to 13 feet?

SOLUTION The change in the area $A(r) = \pi r^2$ when the radius is increased from 1 to 5 feet is

$$A(5) - A(1) = \pi(5^2) - \pi(1^2) = \pi(25 - 1) = 24\pi \text{ square feet.} \quad \textbf{(2)}$$

The change in the area when the radius is increased from from 12 to 13 feet is

$$A(13) - A(12) = \pi(13^2) - \pi(12^2) = (169 - 144)\pi = 25\pi \text{ square feet.} \quad \textbf{(3)}$$

The area is increased more when the radius is increased from 12 to 13 feet because **(3)** is greater than **(2)**. □

Question 2 How does doubling the radius affect the area of a circle?

[†]Answer the questions in the text as you study to check your understanding of concepts. Outlines of responses are given at the end of the section.

Example 2 Find a formula for the area A of a circular ring whose inner radius is r and whose outer radius is $r+1$ with $r \geq 0$ (Figure 5). Then sketch its graph in an rA-plane.

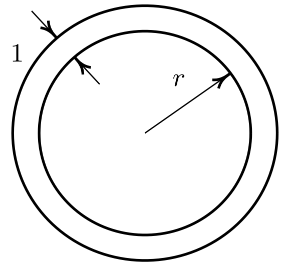


FIGURE 5

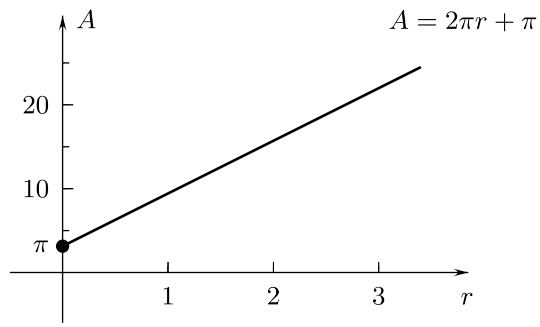


FIGURE 6

SOLUTION The smaller circle in Figure 5 has radius r and the larger circle has radius $r+1$. The area of the ring equals the area of the outer circle minus the areas of the inner circle, which is $\pi(r+1)^2 - \pi r^2 = \pi(r^2+2r+1) - \pi r^2 = 2\pi r + \pi$. The graph of this function is the line $A = 2\pi r + \pi$ for $r \geq 0$ in Figure 6. $\square$.

The function $A = 2\pi r + \pi$ of Figure 6 is called a LINEAR FUNCTION because its graph is part of a straight line.

Example 3 A rocket is fired vertically from the ground at time $t = 0$ (seconds) and rises at a constant velocity until the engine shuts off at $t = 1$. With a mathematical model in which air resistance is ignored, the height of the ball above the ground $h(t)$ (feet) at time t is given by the two formulas

$$h(t) = \begin{cases} 12t & \text{for} \quad 0 \leq t \leq 1 \\ -16t^2 + 44t - 16 & \text{for} \quad 1 < t \leq T. \end{cases} \tag{4}$$

Here T is the time when the rocket hits the ground. The graph of $y = h(t)$ is shown in Figure 7. The portion for $0 \leq t \leq 1$ is formed by the line, $y = 12t$ and the portion for $1 < t \leq T$ is formed by the parabola $y = -16t^2 + 44t - 16$. Use the quadratic formula to find T.

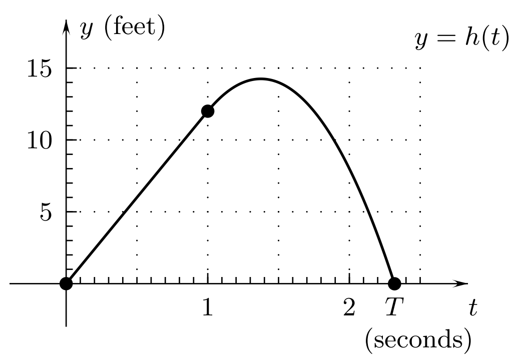


FIGURE 7

SOLUTION Because $h(t) = -16t^2 + 44t - 16$ for $t > 1$, the time T is a solution of $-16T^2 + 44T - 16 = 0$. We divide by -4 to have $4T^2 - 11T + 4 = 0$. Then we use the quadractic formula $T = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2b}$ for the solutions of $aT^2 + btT + c = 0$ and obtain

$$T = \frac{-(-11) \pm \sqrt{(-11)^2 - 4(4)(4)}}{2(4)} = \tfrac{1}{8}(11 \pm \sqrt{57}). \qquad \textbf{(5)}$$

The solution **(5)** has the approximate value 0.431271 with the minus sign and 2.31873 with the plus sign. Since $T > 1$, we use the plus sign and have $T = \frac{1}{8}(11 + \sqrt{57})$. □

Domains of functions given by single formulas

If a function is given by a single formula and no domain is specified or implied from the context in which the function is being used, its domain is assumed to be the set of all values of the variable for which the formula is defined.

Example 4 What are the domain and range of the function $y = \sqrt{x^2 - 1}$ of Figure 8?

SOLUTION The domain of $y = \sqrt{x^2 - 1}$ consists of those numbers x such that $\sqrt{x^2 - 1}$ is defined. This is where $x^2 \geq 1$, which means that $x \leq -1$ or $x \geq 1$. Consequently, the domain consists of the interval $x \leq -1$ and the interval $x \geq 1$. The range is the interval $y \geq 0$ because $y = \sqrt{x^2 - 1}$ takes on all nonnegative numbers y as x varies over the interval $x \leq -1$ and as x varies over the interval $x \geq 1$. □

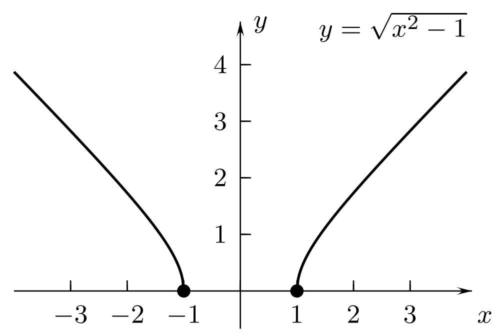


FIGURE 8

Functions given by tables of values

The next function is defined by Table 1, which gives the amount of corn grown by U. S. farmers each year from 1991 through 2000.[1] We let $B(t)$ denote the number of billions of bushels produced in year t, so that $B(1991) = 7.5, B(1992) = 9.5$, etc.

TABLE 1. CORN PRODUCTION BY U.S. FARMERS (BILLIONS OF BUSHELS)

t	1991	1992	1993	1994	1995	1996	1997	1998	1999	2000
$B(t)$	7.5	9.5	6.3	10.0	7.4	9.2	9.2	9.8	9.4	10.0

Figure 9 is a bar graph of the data in Table 1. To obtain the graph of the function $B = B(t)$, we replace each bar with a dot at its top, as in Figure 10. Notice that we use the letter "B" as the name of the function and as the variable on the vertical axis in Figure 10, where we label the graph $B = B(t)$.

[1]Data from *Agricultural Statistics, 2001*, Washington, DC: United United States Government Printing Office, 2001, p. 1-24.

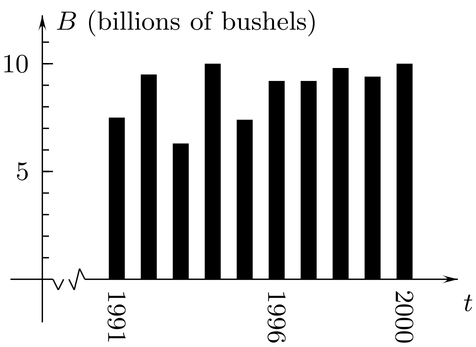


FIGURE 9

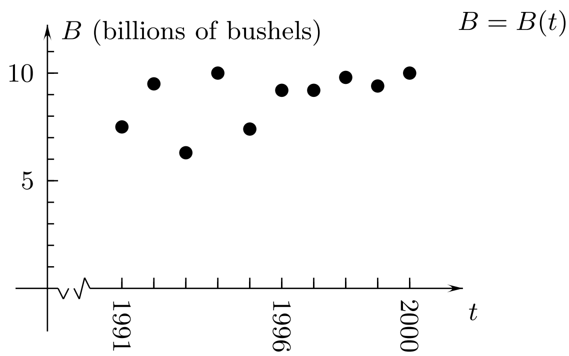


FIGURE 10

Example 5 **(a)** In what years from 1991 through 2000 was the most corn produced in the U.S.? How much was produced in those years? **(b)** For what value of t with $1991 \leq t \leq 2000$ does $B = B(t)$ have its least value? **(c)** What was the total U. S. production of corn in the years 1991 through 2000?

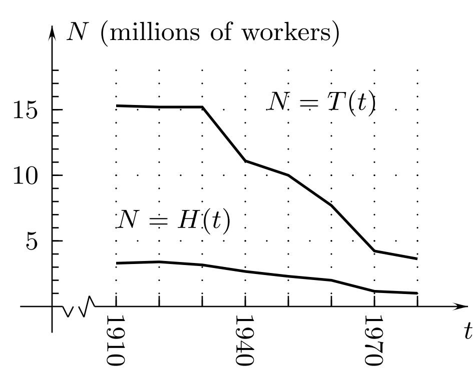
SOLUTION **(a)** According to Table 1, the most corn was produced in 1994 and 2000, when 10 billion bushels were produced.
(b) The least value of $B = B(t)$ in the table is 7.4 billion bushels at $t = 1995$, the year when the least amount of corn was produced.
(c) The total corn production in the years 1991 through 2000 is the sum $7.5 + 9.5 + 6.3 + 10.0 + 7.4 + 9.2 + 9.2 + 9.8 + 9.4 + 10 = 88.3$ (billion bushels) of the numbers in the second row of the table. □

Question 3 Describe one or more ways in which the function $B = B(t)$ of Table 1 is a model that does not represent U. S. corn production exactly.

Functions given approximately by their graphs

The curves in Figure 11 give approximate values of two functions. Their values for $1910 \leq t \leq 1980$ were determined from census data for $1910, 1920, 1930, \ldots, 1980$ by plotting the data and joining successive points with line segments.[2] This process is called LINEAR INTERPOLATION. In this study, farm workers were divided into two categories: those who were members of the families who owned the farms and others who were hired. $N = T(t)$ gives the total number of farm workers and $N = H(t)$ gives the number of those that were hired in year t in the United States. The values of both functions are measured in millions of workers.

FIGURE 11



(2) Data adapted from "Labor-intensive Agriculture" by P. Martin, *Scientific American*, New York, NY: Scientific American, Inc., October, 1983, p. 56.

Example 6 Approximately what percentage of U. S. farm workers were hired **(a)** in 1910 and **(b)** in 1980?

SOLUTION **(a)** From Figure 11 we see that $T(1910) \approx 15.3$ and $H(1910) \approx 3.3$. The fraction of hired farm workers in 1910 was $\dfrac{H(1910)}{T(1910)} \approx \dfrac{3.3}{15.3} \doteq 0.22$, so that in 1910 about 22% of farm workers were hired.

(b) Since $T(1980) \approx 3.6$ and $H(1980) \approx 1$, the fraction of hired workers in 1980 was $\dfrac{H(1980)}{T(1980} \approx \dfrac{1}{3.3} \doteq 0.30$. In 1980 approximately 30% of farm workers were hired. □

Finding formulas for mathematical models

One way to obtain a mathematical model is to find a formula that fits or approximately fits empirical data. The next table gives the velocity v (feet per second) of flame through a hydrogen-air mixture as a function of the percent P of hydrogen in the mixture for $P = 5, 15, 25, \ldots, 65$.[3] This data is plotted in Figure 12. The flame velocity is relatively small for $P = 5$ because there is not much hydrogen to burn. The velocity is also relatively small for $P = 65$ because in this case there is not much oxygen to combine with the hydrogen in the burning process.

TABLE 2. FLAME VELOCITY

P (percent)	5	15	25	35	45	55	65
v (feet per second)	0.8	5.0	11.6	15.7	15.0	8.5	2.9

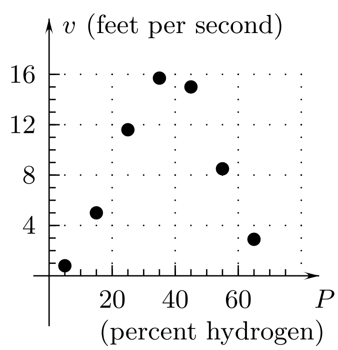


FIGURE 12

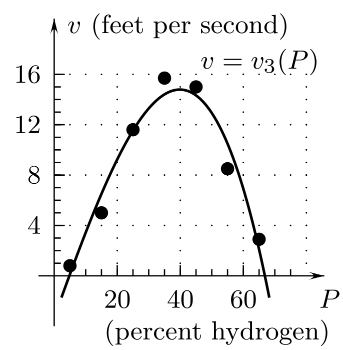


FIGURE 13

If we wanted a formula for calculating approximate flame velocities from percents of hydrogen without referring to the table, we could use a procedure on a graphing calculator or in calculus computer software to find a polynomial that approximates the data in the table. Figure 13 shows the given data with the graph of an approximating third-degree polynomial,

$$v_3(P) = -(1.33 \times 10^{-4})P^3 - (7.74 \times 10^{-4})P^2 + 0.70P - 3.33 \quad \textbf{(6)}$$

whose coefficients were found with such a procedure on a graphing calculator.

[3] Data adapted from *Handbook of Engineering Materials*, D. Miner and J. Seastone editors, New York, NY: John Wiley & Sons, 1955, p. 3-322.

Example 7 **(a)** What is the largest (positive) error that is made if $v = v_3(P)$ is used to approximate the seven values of v in Table 2?

SOLUTION Figure 13 shows that the largest error using v_3 occurs either at $P = 15, 35$, or 55. We use **(6)** to calculate the values $v_3(15) \doteq 6.5, v_3(35) \doteq 14.5$, and $v_3(55) \doteq 10.7$ of v_3 at the three points and then find the absolute values of the differences between these values of v_3 and the corresponding numbers in Table 2. We obtain

$$|v_3(15) - 5.0| \doteq |6.5 - 5.0| = 1.5$$
$$|v_3(35) - 15.7| \doteq |14.5 - 15.7| = |-1.2| = 1.2$$
$$|v_3(55) - 8.5| \doteq |10.7 - 8.5| = 2.2.$$

The greatest error using v_3 is approximately 2.2 at $P = 55$. □

Question 4 Explain why v_3 is not a good model of the flame velocity for P close to zero or for $P \geq 70$.

When is a graph the graph of a function?

The circle of radius 1 with its center at the origin in Figure 14 is the graph of the equation $x^2 + y^2 = 1$. To determine whether it is the graph of a function, we apply the folowing rule.

Rule 1 (The vertical line test) *A nonempty set of points in the xy-plane is the graph of a function if and only if no vertical line intersects it at more than one point. In this case the domain of the function consists of the x-coordinates of the points in the set and its values are the corresponding y-coordinates.*

This rule applies because a function cannot have more than one value at one point.

The circle in Figure 14 is not the graph of a function because the vertical line at any x with $-1 < x < 1$, as in Figure 14, intersects the circle at two points.

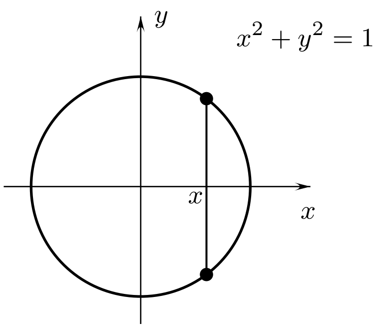


FIGURE 14

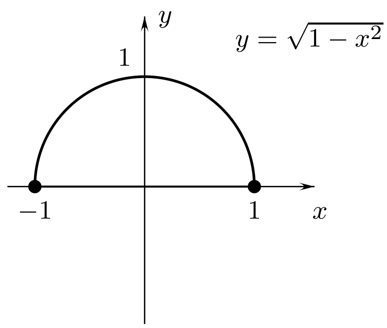


FIGURE 15

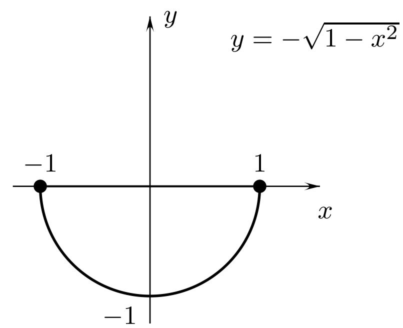


FIGURE 16

To use functions to study the circle $x^2 + y^2 = 1$ we solve for y. We write $y^2 = 1 - x^2$ and then $y = \pm\sqrt{1 - x^2}$. This gives two functions $y = \sqrt{1 - x^2}$ and $y = -\sqrt{1 - x^2}$, whose graphs are the semicircles in Figures 15 and 16.

Example 8 Figures 17 and 18 show two curves, one of which is the graph of a function $y = f(x)$. Which curve is that graph and what are the domain and range of the function?

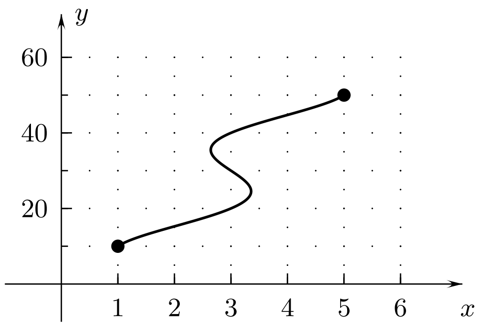


FIGURE 17

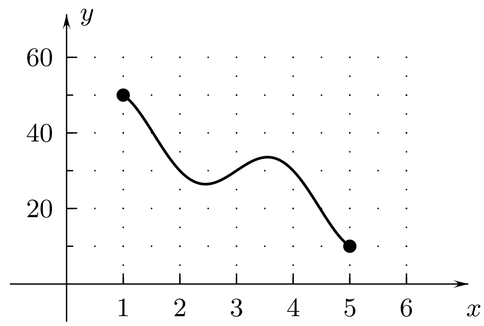


FIGURE 18

SOLUTION The curve in Figure 17 is not the graph of a function $y = f(x)$ because some vertical lines, such as the line $x = 3$ in Figure 19, intersect it in three points and a function cannot have three values at the one value of x.

The curve in Figure 18 is the graph of a function $y = f(x)$. The domain of this function is the interval $1 \leq x \leq 5$ on the x-axis and its range is the interval $10 \leq y \leq 50$ on the y-axis, as indicated in Figure 20. □

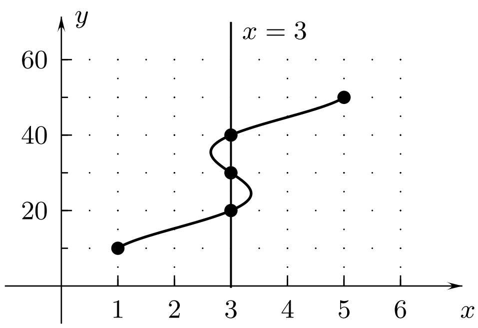


FIGURE 19

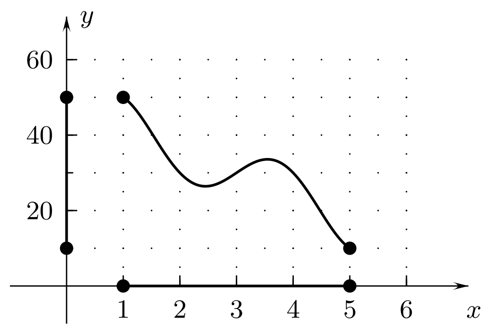


FIGURE 20

Percent and relative change of values of functions

Changes in quantities are often described by relative or percent changes. Suppose for example, that the price of a TV set is raised from \$300 to \$330. We subtract the old price of \$300 from the new price of \$330 to get the change in price, which is \$30. Then we divide by the old price to obtain \$30/\$300 = 0.10. This is the fraction or proportion by which the price is raised and is known as the RELATIVE CHANGE. Multiplying the relative change by 100 converts it to the PERCENT CHANGE, which is 10%. These calculations can be made in any context:

Rule 2 (Relative and percent change) *For any numbers A and B with $A > 0$,*

$$[\textit{The change from } A \textit{ to } B] = B - A$$

$$[\textit{The relative change from } A \textit{ to } B] = \frac{B - A}{A}$$

$$[\textit{The percent change from } A \textit{ to } B] = \frac{100(B - A)}{A}.$$

Example 9 **(a)** What are the relative and percent changes from 10 to 13? **(b)** What are the relative and percent changes from 100 to 103? **(c)** What are the relative and percent changes from 100 to 95?

SOLUTION **(a)** The change from 10 to 13 is 3, so the relative change is $\frac{3}{10} = 0.3$ and the percent change is 30%.
(b) The change from 100 to 103 is 3, so the relative change is $\frac{3}{100} = 0.03$ and the percent change is 3%.
(c) The change from 100 to 95 is -5, so the relative change is -0.05 and the percent change is -5%. □

When the formulas in Rule 2 are applied to values of a function $y = f(x)$, they read as follows:

$$[\text{The change of } f(x) \text{ from } x_1 \text{ to } x_2] = f(x_2) - f(x_1)$$

$$[\text{The relative change of } f(x) \text{ from } x_1 \text{ to } x_2] = \frac{f(x_2) - f(x_1)}{f(x_1)}$$

$$[\text{The percent change of } f(x) \text{ from } x_1 \text{ to } x_2] = 100\left[\frac{f(x_2) - f(x_1)}{f(x_1)}\right].$$

Question 5 What are the change, relative change, and percent change in $f(x) = x^3$ when x goes from 2 to 3?

Responses 0.1

Response 1 The dot at the origin would be replaced by a small circle, as in Figure R1.

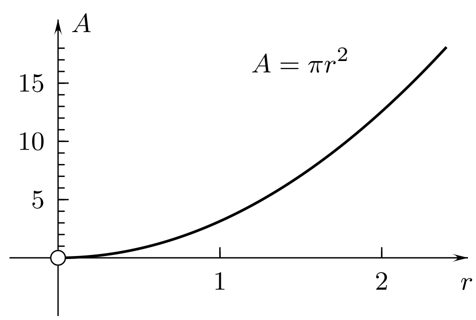


Figure R1

Response 2 The area of a circle of radius $2r$ is $A(2r) = \pi(2r)^2 = 4\pi r^2 = 4A(r)$. • Doubling the radius quadruples the area.

Response 3 Possible answers: There is no telling how carefully the bushels were measured, and the results have been rounded off to the nearest 100 million bushels.

Response 4 $v_3(P)$ is negative for small positive P and for $P > 70$, and the flame velocity cannot be negative.

Response 5 $f(2) = 2^3 = 8$ • $f(3) = 3^3 = 27$ • [Change in $f(x)$ from $x = 2$ to $x = 3$] $= 27 - 8 = 19$ • [Relative change] $= \frac{19}{8}$ • [Percent change] $= \frac{19}{8}(100) = 237.5\%$

Interactive Examples 0.1

Interactive solutions are on the web page http//www.math.ucsd.edu/˜ashenk/.[†]

1. What is $K(-2)$ if

$$K(t) = \begin{cases} 2 + 1/t & \text{for} \quad -3 < t < 0 \\ 5 & \text{for} \quad t = 0 \\ -t^2 & \text{for} \quad t > 0? \end{cases}$$

2. **(a)** Give approximate values of $A(1)$ and $A(4)$ for the function $y = A(x)$ of Figure 21. **(b)** Find the approximate solutions x of $A(x) = 40$ with $0 \le x \le 6$.

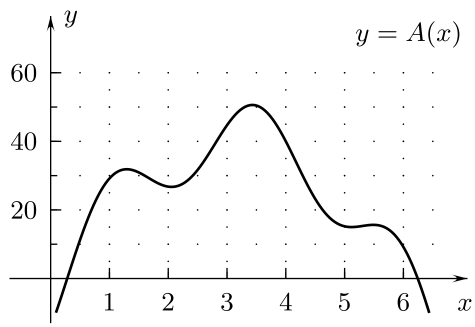


FIGURE 21

3. What is the domain of $y = \dfrac{\sqrt{x}}{x-2}$?

4. The domain of the function $y = G(x)$ consists of the integers $x = 1, 2, 3$, and 4. Its values satisfy the equations $G(1) = 2, G(2) = 3G(1) + 1, G(1) + G(2) + G(3) = 0$, and $G(4) = 2G(3)$. Find its four values and sketch its graph.

Exercises 0.1

[A]Answer provided. [O]Outline of solution provided. [C]Graphing calculator or computer required.

CONCEPTS:

1. The symbols $A(2)$ have one meaning if A is a general function $y = A(x)$ and another meaning if A represents a constant. Explain.

2. What are $p(1), p(2)$, and $p(3)$ if the points $(1, 18)$, $(2, -6)$, and $(3, \sqrt{2})$ are on the graph of p?

3. List in order the steps—adding, multiplying, and finding values of g—that are used to find **(a)** $g(a + b)$, **(b)** $g(a) + g(b)$, **(c)** $g(ab)$, and **(d)** $g(a)g(b)$ from a and b.

4. **(a)** Explain why the five points $(1, 4), (5, 15), (5, 0), (6, 4)$, and $(10, 3)$ do not form the graph of a function. **(b)** Pick four points from part (a) that do form the graph of a function and describe its domain and range.

5. What must be the value of the constant k in order that the definition $f(x) = \begin{cases} 2x & \text{for} \quad x \le 1 \\ k & \text{for} \quad x \ge 1 \end{cases}$ make sense?

6. Give examples from everyday speech where a change is referred to as "large" or "small" depending on whether a relative change is large or small. (As one example, you might compare inviting ten extra people to a dinner party with increasing the population of a city by ten.)

[†]In the published text the interactive solutions of these examples will be on an accompanying CD disk which can be run by any computer browser without using an internet connection.

BASICS:

7

7^{O} Give the values of $y = \dfrac{(x-3)(x+4)}{(x+2)(x-1)(x+5)}$ at **(a)** $x = -3$, **(b)** $x = 0$, and **(c)** $x = 3$. **(d)** At what values of x is this function not defined?

$8.^{O}$ What are $f(0), f(2), f(-5)$, and $f(4)$ for $f(x) = \dfrac{x^2}{\sqrt{5-2x}}$?

$9.^{O}$ Give a formula without parentheses for $F(-x)$ where $F(t) = \dfrac{1-t^2+3t^3}{10-t}$.

$10.^{O}$ Compare $Q(t)$ and $Q(1/t)$ where $Q(x) = x + \dfrac{1}{x} + 2$.

$11.^{O}$ Give a formula without parentheses for $W(2+a) - W(2)$ where $W(t) = 5t + 3t^2$.

$12.^{O}$ What are **(a)** the change, **(b)** the relative change, and **(c)** the percent change from 54 pounds to 60 pounds?

$13.^{O}$ What are **(a)** the change, **(b)** the relative change, and **(c)** the percent change in $y = 2^x$ from $x = 2$ to $x = 5$?

$14.^{O}$ **(a)** Which of the curves in Figures 22 and 23 is the graph of a function $y = g(x)$? **(b)** What are the domain and range of this function?

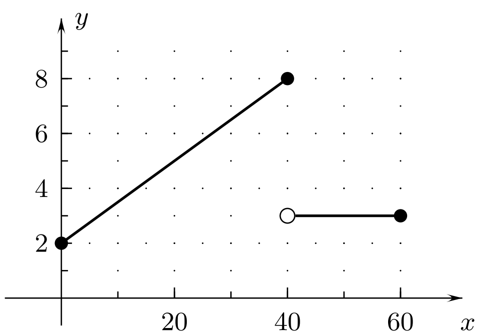


FIGURE 22

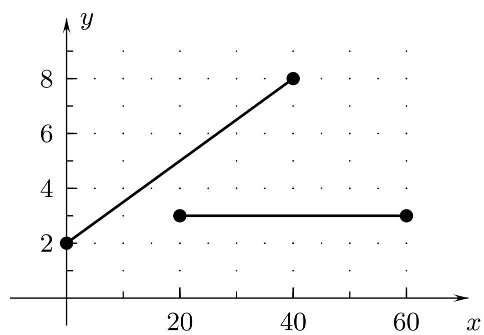


FIGURE 23

$15.^{O}$ Give **(a)** the approximate values of $B(-2), -2B(-1)$, and $[B(2)]^2$ and **(b)** the approximate solutions u of $B(u) = 1$ for the function B of Figure 24.

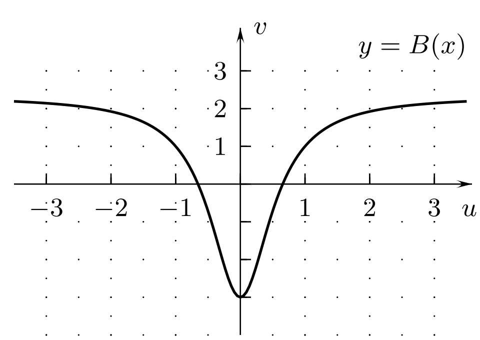


FIGURE 24

16.[A] Figure 25 shows the graph of the average credit card debt per household $y = D(t)$ (thousand dollars) as a function of the year t.[(4)] **(a)** What was the approximate change in the average credit card debt per household from the beginning of 1990 to the beginning of 2000? **(b)** When was the average credit card debt approximately half what it was at the beginning of 2000?

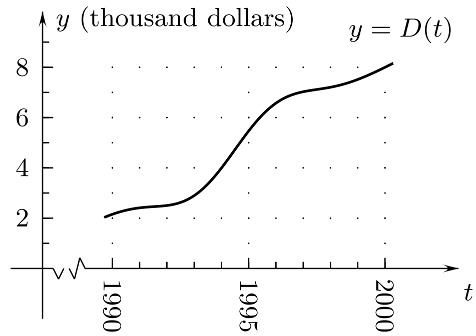


FIGURE 25

17. If an unclothed person is in a room at a constant temperature T_A (the ambient temperature) for several hours, his or her body core will reach a constant temperature T_B. Figure 26 shows the graph of the body core temperature as a function $T_B = f(T_A)$ of the ambient temperature.[(5)] In approximately what interval of ambient temperatures can the body can keep its core temperature within a few degrees of the normal 98.6°F?

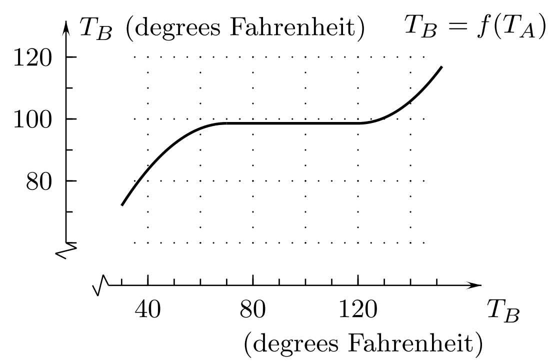


FIGURE 26

18.[A] Figure 27 is a bar graph of the number of national savings and loan failures $N = N(t)$ in year t from 1980 through 1990.[(6)] **(a)** What was the approximate greatest decrease in $N(t)$ from one year to the next during this time period? **(b)** How many years did it take for $N(t)$ to go from its minimum value to its maximum value?

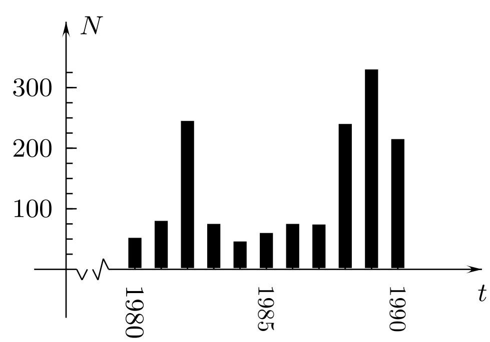


FIGURE 27

[(4)]Data adapted from *Newsweek*, August 27, 2001, p. 36

[(5)]Data adapted from *Textbook of Medical Physiology* by A. Guyton and J. Hall, Philadelphia: W. B. Saunders, Co., 2000, p. 826.

[(6)]Data adapted from *Los Angeles Times*, Los Angeles, CA: The Times Mirror Company, January 2, 1991. Sources: Office of Thrift Supervision and the Resolution Trust Corp.

19. The table below lists average monthly rainfall $y = F(t)$ in the country of Belize and $y = G(t)$ in the city of Beirut, Lebanon, as functions of the month, with $t = 1$ for January, $t = 2$ for February, etc.[7]
(a) What are the values of $F(3)$ and $G(3)$ and what do they represent? **(b)** What is the relative change in F from $t = 3$ to $t = 7$? **(c)** What is the percent change in G from $t = 3$ to $t = 7$?

AVERAGE MONTHLY RAINFALL (INCHES)

MONTH	t	$F(t)$	$G(t)$	MONTH	t	$F(t)$	$G(t)$
January	1	5.4	7.5	July	7	6.4	0
February	2	2.4	6.2	August	8	6.7	0
March	3	1.5	3.7	September	9	9.6	0.2
April	4	2.2	2.2	October	10	12.0	2.0
May	5	4.3	0.7	November	11	8.9	5.2
June	6	7.7	0.1	December	12	7.3	7.3

20.[A] **(a)** What is the domain of $y = x^2 + 1/x^2$? [C]**(b)** Generate the graph in the window $--3 \le x \le 3$, $-1 \le y \le 9$ and use it to find the approximate range of the function.

21. **(a)** What is the domain of $y = \sqrt{16 - x^4}$? [C]**(b)** Generate the graph of $y = \sqrt{16 - x^4}$ in the window $-3 \le x \le 3, -1 \le y \le 5$ and use it to find the range of the function.

22. **(a)** What are the domains of **(a[A])** $y = \sqrt{x+1}$, **(b)** $y = \sqrt[3]{x-1}$, and **(c)** $y = \dfrac{1-5x^2}{5x^2}$?
[C]**(d)** Generate the graphs separately in the window $-2 \le x \le 3$, $-2 \le y \le 2$ as a partial check of your answers.

23.[O] Which is the greatest and which is the least of the numbers $f(1), f(2)$, and $f(3)$ if $f(x) = 4x - 4x^2 + x^3$?

24.[A] Suppose that $A(x) = \sqrt{x}, B(x) = A(x+5)$, and $C(x) = A(x) + 5$. Which is greater $B(4)$ or $C(4)$?

25. Which is greater $P(2) + P(5)$ or $P(2+5)$ if $P(x) = x^2$?

26. Calculate approximate decimal values of **(a[A])** $f(x) = x + 4x^3 - 5x^5 + x^6$ at $x = -1.2543$, **(b[A])** $g(x) = \dfrac{x^{1.6} - 6}{7 - x^{1.7}}$ at $x = 4.17$, **(c)** $h(x) = \left(\dfrac{x^2+6}{x^2+3}\right)^5$ at $x = 1092$, and **(d)** $k(x) = \sqrt[10]{10^x + x^{10}}$ at $x = 35$.

27.[A] Give the values of **(a)** $F(10)$, **(b)** $G(3)$, **(c)** $H(2)$, and **(d)** $\sqrt{F(2)}$ where $F(x) = x^4 - 3x^2 + 5$, $G(x) = xF(x)$, and $H(x) = F(\sqrt{x})$.

28.[A] The balance after one year on a deposit of \$1000 that earns $r\%$ annual interest compounded semiannually is $B(r) = 1000 + 10r + \frac{1}{40}r^2$ dollars. **(a)** What is the balance after one year if the interest rate is 8%? **(b)** If the interest rate is 10%? **(c)** Show that $B(r) = 1000(1 + \frac{1}{200}r)^2$.

29. The density ρ (rho) of dry air at a pressure of one atmosphere and a temperature of T degrees Celsius is $\rho(T) = \dfrac{0.001293}{1 + (0.00367)T}$ grams per milliliter.[8] What is its density **(a)** at 10° Celsius and **(b)** at 50° Celsius? **(c)** Does the density increase or decrease as the temperature increases?

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[7] Data from *Fodor's World Weather Guide* by E. Pierce and C. Smith, New York: Random House, 1984, pp. 62 and 215.
[8] *CRC Handbook of Chemistry and Physics*, 62nd Edition, Boca Raton, FL: CRC Press, 1981, p. F-11.

30.$^{\mathsf{A}}$ The graph of a function $y = W(x)$ is formed by joining the successive points $(1,1)$, $(2,5)$, $(3,2)$, and $(5,2)$ in an xy-plane by straight lines. **(a)** What is the domain of $y = W(x)$? **(b)** Draw its graph. **(c)** What are $W(1), W(3), W(4)$, and $W(4.5)$?

EXPLORATION:

31. What are the domains of **(a**$^{\mathsf{A}}$**)** $y = (1 + x^5)^{-1/3}$, **(b)** $y = \dfrac{1}{x^2 - 4x}$, **(c)** $y = \sqrt{x+2} - \sqrt{3-x}$, **(d**$^{\mathsf{A}}$**)** $y = \begin{cases} x^2 & \text{for } x < 2 \\ 4x & \text{for } 2 < x \leq 10 \end{cases}$, and **(e)** $y = (2+x)^{-1/2}$?

32. The table below gives the number $N(t)$ (millions) of suburban utility vehicles (SUV's) that were sold in the Unitied States in the even years t from 1990 through 1999.[9] Express each of the following statements as English sentences without using the mathematical symbols t or N. Then determine from the table whether the statement is true or false. **(a**$^{\mathsf{A}}$**)** $N(1999) > 3N(1990)$, **(b)** $N(1996) > N(1990) + 2$, and **(c)** $N(1996) = N(1990) + N(1992) + N(1994)$.

SUV SALES IN THE UNITED STATES

t (year)	1990	1992	1994	1996	1998	1999
$N(t)$ (millions)	1.0	1.1	1.6	2.1	2.8	3.3

33. The next table gives the grade-point-average $G = G(t)$ in undergraduate courses in the spring quarter at the University of California, Los Angeles, as a function of the year t.[10] Express each of the following statements as mathematical statements, using no English words. Then determine from the table whether the statement is true or false. **(a**$^{\mathsf{A}}$**)** The grade-point average was more than half a point higher in 1999 than in 1949. **(b)** The grade-point average was higher in 1979 than in 1969. **(c**$^{\mathsf{A}}$**)** The grade-point average increased by more than 20% from 1949 to 1999.

GRADE-POINT AVERAGES AT UCLA

t (year)	1949	1959	1969	1979	1990	1999
$G(t)$	2.40	2.60	2.80	2.78	2.88	3.02

34.$^{\mathsf{A}}$ If the radius of a right circular cone is r feet and it is h feet high, then the total surface area of its base and lateral surface is $\pi r^2 + \pi r\sqrt{r^2 + h^2}$ square feet. **(a)** Which part of this formula is the area of the base? **(b)** What is the total surface area if $r = 2$ and $h = 3$? **(c)** Give formulas for the total surface area $A(h)$ in terms of h for $r = 1, 2,$ and 3. $^{\mathsf{C}}$**(c)** Generate the graphs of the three functions from part (c) together in the window $0 \leq h \leq 10, 0 \leq A \leq 130$ and copy them on your paper.† Describe how increasing r changes the shape of the graph.

35. What are the domains of **(a)** $y = f(x^3)$ and **(b)** $y = [f(x)]^3$ if the domain of f is the interval $0 \leq x \leq 8$?

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[9] Data adapted from *Newsweek*, Source: Ward's Automobile Information Bank, July 2, 2001, p. 42.

[10] Data adapted from the *UCSD Faculty Newsletter*, Source: The University of California Office of Planning and Budget, December 3, 1999.

† Whenever you are asked in this text to generate a curve, generate it on your calculator or computer.

36. A small potato at room temperature was put in an oven then taken out and left to cool. The graph of its temperature as a function of time is shown in Figure 28. **(a)** What was the approximate temperature of the room? **(b)** The temperature of the oven was either 120° F, 190° F, or 350° F. Which is possible? Explain. **(c)** When was the potato taken out of the oven? **(d)** Suppose a much larger potato is used. Trace the curve in Figure 21 and sketch in the same drawing a plausible graph of the temperature of the larger potato. Explain how and why the two graphs differ.

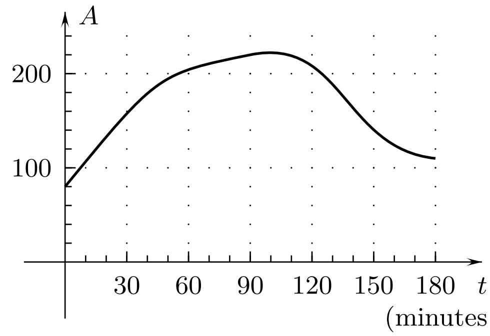


FIGURE 28

37.[A] The world's human population at the beginning of each century from 100 AD to 2000 AD is given below, with the numbers rounded to the nearest 10 million.[11] Figure 29 is a bar graph of the population. **(a)** Calculate the changes and relative changes in population from 100 AD to 1000 AD, from 1000 AD to 1800 AD, and from 1800 AD to 2000 AD. **(b)** Find the change and relative change in population from 100 AD to 2000 AD. **(c)** One of the answers in part (b) is a sum of answers from part (a) and the other is not. Which is which? **(d)** In which century did the population decrease and why?

World population (millions)

Year A.D.	100	200	300	400	500	600	700	800	900	1000
Population	180	190	190	190	190	200	210	220	230	240

Year A.D.	1100	1200	1300	1400	1500	1600	1700	1800	1900	2000
Population	300	360	360	350	415	545	610	900	1630	6150

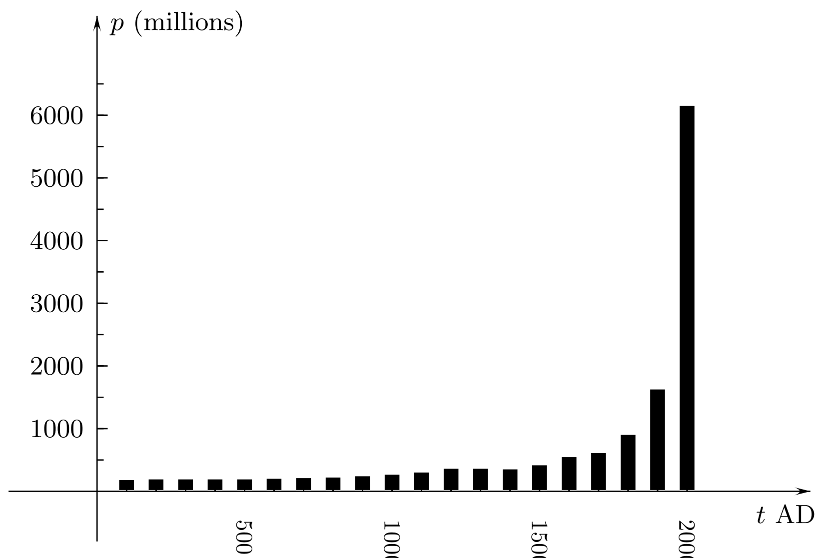


FIGURE 29

[11] Data adapted from *World Health and Disease* by A. Gray, Buckingham: Open University Press, 1993, p. 49.

38. Use your calculator or computer to find the approximate decimal value of $f(f(f(f(f(10)))))$ where $f(x) = \sqrt{x}$.

39. The upper curve labeled E in Figure 30 was the source of the data on flame velocity in hydrogen/air mixtures that was used in Example 7. Curve C gives the flame velocities for propane/air mixtures and curve D gives the flame velocities for carbon monoxide/air mixtures. **(a)** How does the maximum flame velocity for hydrogen/air mixtures compare with the maximum flame velocity for carbon monoxide/air and propane/air mixtures? ($^{\mathrm{C}}$**b)** use a procedure on calculator or computer to find a quadratic polynomial $v(P) = aP^2 + bP + c$ that approximates the data for hydrogen/air mixtures.

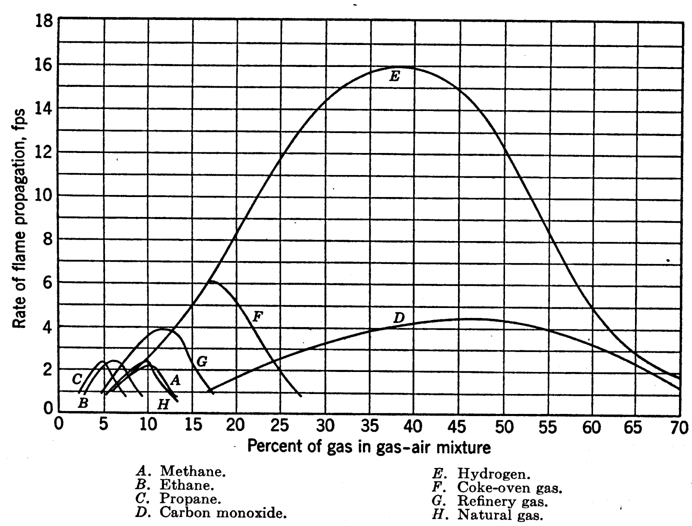


FIGURE 30

40. One square is circumscribed about a circle of radius 1, and another square is inscribed in the circle (Figure 31). **(a)** What percent of the area of the circumscribed square is not in the circle? **(b)** What percent of the area of the circle is not in the inscribed square? **(c)** How are the areas of the two squares related?

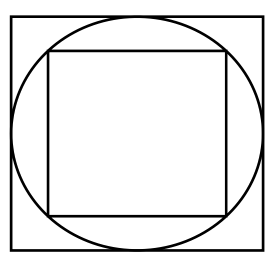


FIGURE 31

41. The next table gives the average number of additional years $L = L(t)$ that U.S. residents at various ages t could expect to live at the beginning of 1999.[12] **(a)** What does $D(t) = t + L(t)$ represent? **(b)** How much greater is $D(80)$ than $D(0)$ and what does this mean?

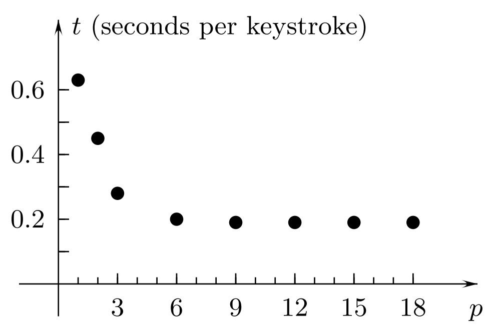
ADDITIONAL LIFE EXPECTANCY AS A FUNCTION OF AGE

t	0	10	20	30	40	50	60	70	80
$L(t)$	76.7	67.4	57.7	48.2	38.8	29.8	21.5	14.3	8.5

42. The following table gives results from experiments run to determine to what extent expert typists look ahead to characters they have not yet reached as they type.[13] Each typist in the study read a screen which showed a fixed number p of characters beyond the one he or she was typing. The typing speeds were averaged to obtain the values of $t = t(p)$ in the second row of the table. For instance, $t(2) = 0.450$ because it took the typists an average of 0.450 seconds to type a character when they were shown two characters ahead of the one they were typing. Figure 32 is a graph of t as a function of p. **(a)** Why is it plausible that $t(2) < t(1)$ and $t(3) < t(2)$? **(b)** Why is it plausible that $t(6), t(9), t(12)$, $t(15)$, and $t(18)$ are approximately equal? **(c)** What would be the likely value of $t(25)$? Why? **(d)** Suppose you could only use the table or the graph of typing speeds in a presentation. What would be some advantages and disadvantages of each method of displaying the data?

TYPING SPEEDS

p (characters seen ahead)	1	2	3	6	9	12	15	18
$t(p)$ (seconds per character)	0.620	0.450	0.280	0.20	0.191	0.189	0.190	0.192



$t = t(p)$

FIGURE 32

(End of Section 0.1)

[12] Data from *Statistical Abstract of the United States*, Washington, DC: U.S. Department of Commerce, 203, p.73.

[13] Data from "The Skill of Typing" by Timothy A. Salthouse, *Scientific American*, February, 1984, p. 129.