Section 0.2

(5/31/07)

Set notation and solving inequalities

OVERVIEW: *Inequalities are almost as important as equations in calculus. Many functions' domains are intervals, which are defined by inequalities. Inequalities are needed to study where functions have positive and negative values. They are also used in the definitions of limits and with derivatives to study where functions are increasing and decreasing and where their graphs are concave up and concave down. In this section we describe notation and terminoogy for intervals and other sets and discuss the rules for solving inequalities. These rules are similar to those for solving equations but are somewhat more difficult to apply.*

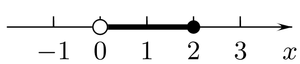
Topics:

- ***Intervals and other sets of numbers***
- ***The absolute value function***
- ***Working with inequalities***

Intervals and other sets of numbers

Intervals can be defined, as in the last section, by giving their defining inequalities. With this approach the interval in Figure 1 is called the interval $0 < x \le 2$. The heavy line in the drawing indicates that the points x with $0 < x < 2$ are in the interval; the dot shows that $x = 2$ is in the interval; and the open circle at $x = 0$ indicates that the point $x = 0$ is not.

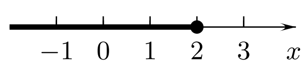
We can also define intervals with SET-BUILDER NOTATION: The symbols $\{x : P\}$ designate the set of numbers x that satisfy condition P. With this notation the interval in Figure 1 can be defined by $\{x : 0 < x \le 2\}$, which reads "the set of those numbers x such that $0 < x \le 2$." We also refer to this interval as $(0, 2]$, where the parenthesis at the left indicates that the point $x = 0$ is not in the interval and the square bracket at the right indicates that the point $x = 2$ is in the interval. Similarly, the interval in Figure 2 can be referred to either as the interval $x \le 2$, as the interval $\{x : x \le 2\}$, or as the interval $(-\infty, 2]$; and the interval in Figure 3 can be given by $x > 1$, by $\{x : x > 1\}$, or by $(1, \infty)$.



The interval $0 < x \le 2$ or

$\{x : 0 < x \le 2\} = (0, 2]$

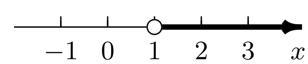
FIGURE 1



The interval $x \le 2$ or

$\{x : x \le 2\} = (-\infty, 2]$

FIGURE 2



The interval $x > 1$ or

$\{x : x > 1\} = (1, \infty)$

FIGURE 3

To describe a set that consists of two intervals, we use the UNION symbol $\cup$ with the convention that $A \cup B$ designates the set consisting of the points in set A combined with the points in set B:†

$$A \cup B = \{x : x \text{ is in } A \text{ or } x \text{ is in } B\}.$$

Example 1 **(a)** Draw on an x-axis the set of points $\{x : -5 \le x < 2 \text{ or } x > 4\}$.
(b) Express the set in part (a) as a union of intervals.

SOLUTION **(a)** We show the intervals on an x-axis by drawing solid lines from $x = -5$ to $x = 2$ and to the right of $x = 4$ and by putting a solid dot at $x = -5$ and small open circles at $x = 2$ and $x = 4$, as in Figure 4.
(b) Since $\{x : -5 \le x < 2 \text{ or } x > 4\}$ consists of the interval $-5 \le x < 2$ and the interval $x > 4$, it is their union $[-5, 2) \cup (4, \infty)$. □

† The symbol $\cap$ is used for the INTERSECTION $A \cap B = \{x : x \text{ is in } A \text{ and } x \text{ is in } B\}$ of two sets A and B.

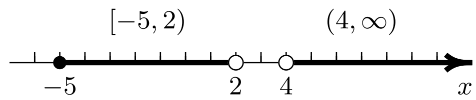


FIGURE 4

The absolute value function

When we want to consider the size of a number x without reference to whether it is positive or negative, we use its ABSOLUTE VALUE, denoted $|x|$. The absolute value of x equals x if x is ≥ 0 and can obtained by multiplying x by -1 if it is negative, so that $|5| = 5$ and $|-5| = -(-5) = 5$. Thus, the absolutevalue function $y = |x|$ with variable x can be defined by

$$|x| = \begin{cases} x & \text{for} \quad x \geq 0 \\ -x & \text{for} \quad x < 0. \end{cases} \tag{1}$$

Definition **(1)** shows that the graph of the absolute value function consists of the line $y = x$ for $x \geq 0$ and the line $y = -x$ for $x < 0$, as shown in Figure 5.

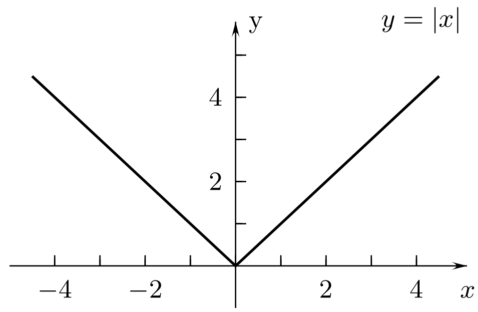


FIGURE 5

The absolute value of a number x can also be obtained with the formula

$$|x| = \sqrt{x^2} \tag{2}$$

since, by convention, $\sqrt{x^2}$ denotes the nonnegative square root of x^2. For example, if $x = -5$, then $\sqrt{(-5)^2} = \sqrt{25} = 5$ and this is the absolute value of -5.

Frequently it is convenient to think of the absolute value $|x|$ of a number x as its distance from the origin on an x-axis (Figure 6). We can also think of $|a - b|$ as the distance between the points a and b on an x-axis (Figure 7).

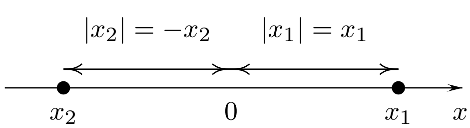


FIGURE 6

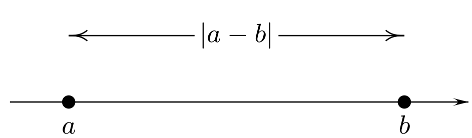


FIGURE 7

Example 2 Solve the equation $|2x - 8| = 6$ for x.

SOLUTION ONE SOLUTION: $|2x-8| = 6$ if $2x - 8 = 6$ or if $2x - 8 = -6$. We solve these equations by adding 8 to both sides of each and then dividing both sides of each by 2:

$$\begin{aligned} 2x - 8 &= 6 \qquad & 2x - 8 &= -6 \\ 2x &= 14 & 2x &= 2 \\ x &= 7 & x &= 1 \end{aligned}$$

The solutions are $x = 7$ and $x = 1$. This is illustrated in Figure 8 by the curve $y = |2x - 8|$ and the line $y = 6$, which intersect at $x = 1$ and $x = 7$.

ALTERNATE SOLUTION: Dividing both sides of $|2x - 8| = 6$ by 2 gives $|x - 4| = 3$, so the solutions are the points $x = 1$ and $x = 7$ that are a distance 3 from $x = 4$, as can be seen in Figure 8. □

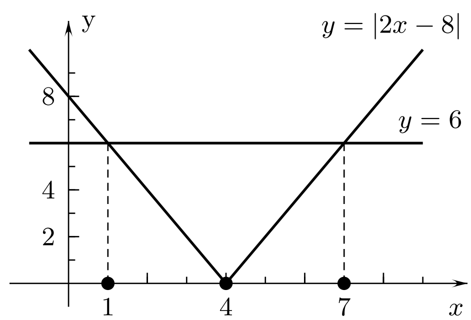


FIGURE 8

Question 1 Verify the solutions in Example 2 by substituting the values of x in the formula $|2x-8|$.

The set consisting of a finite number of real numbers $x_1, x_2, \ldots, x_n$ is denoted $\{x_1, x_2, \ldots, x_n\}$. Thus, the SOLUTION SET (the set of solutions) of the equation in Example 2 is the set $\{1, 7\}$.

Working with inequalities

An INEQUALITY is a statement such as $x^3 - 1 < 4x^3 - 4$ that involves an INEQUALITY SYMBOL $<, >, \leq$, OR $\geq$. Two inequalities involving the variable x are EQUIVALENT if they are satisfied by the same values of x. We say that an inequality has been SOLVED if it has been replaced by an equivalent inequality in which the variable appears alone on one side of the inequality sign and not on the other, as in the statement $x > -1$.

How does solving inequalities differ from solving equations? If we want to solve the equation $x^3 - 1 = 4x^3 + 2$, we can add 1 to both sides to obtain $x^3 = 4x^3 + 3$; subtract $4x$ from both sides to have $-3x^3 = 3$; divide both sides by -3 to have $x^3 = -1$; and finally take cube roots of both sides to obtain the solution $x = -1$. We can also perform such procedures on inequalities, but with some modifications. Here are some basic rules:

Theorem 1 **(a)** *Adding a number to both sides of an inequality, subtracting a number from both sides, or multiplying or dividing both sides by a positive number yields an equivalent inequality.*

(b) *Multiplying or dividing both sides of an inequality by a negative number and reversing the direction of the inequality sign yields an equivalent inequality.*

(c) *Taking an odd power or odd root of both sides of an inequality yields an equivalent inequality.*

(d) *If the numbers on both sides of an inequality are nonnegative, then taking an even power or even root of both sides yields an equivalent inequality.*

Example 3 Find the solution set of the inequality $6 + 2x \le 10$.

SOLUTION Subtracting 6 from both sides of $6 + 2x \le 10$ gives the equivalent inequality $2x \le 4$. Then dividing both sides by the positive number 2 yields $x \le 2$. The solution set is the interval $(-\infty, 2]$. □

C ***Question 2*** Illustrate the result of Example 3 by generating $y = 6 + 2x$ and $y = 10$ together on your calculator or computer in the window $-5 \le x \le 5, -5 \le y \le 15$.†

Example 4 Solve the inequality $x^3 - 1 < 4x^3 + 2$.

SOLUTION Adding 1 to both sides of the inequality gives $x^3 < 4x^3 + 3$. Subtracting $4x^3$ from both sides yields $-3x^2 < 3$. Dividing both sides by the negative number -3 and reversing the direction of the inequality sign gives $x^3 > -1$. Taking cube roots of both sides gives $x > -1$. □

C ***Question 3*** Generate $y = x^3 - 1$ and $y = 4x^3 + 2$ together in the window $-2 \le x \le 2, -8 \le y \le 8$ and use the curves to explain the solution of Example 4.

Example 5 The smallest and largest modern Mexican coins are a centavo coined in the 1970's and a ten peso piece from the 1950's (Figure 9).(1) The radius of the centavo is 0.65 centimeters and the radius of the ten peso piece is 2 centimeters. Consequently, the radii of all modern Mexican coins lie in the interval $0.65 \le r \le 2$ with r measured in centimeters. What is the smallest interval that contains the areas of all modern Mexican coins?

FIGURE 9

SOLUTION Squaring the three positive numbers in the inequalities $0.65 \le r \le 2$ gives the equivalent inequalities $(0.65)^2 \le r^2 \le 2^2$. Then multiplying the new numbers by the positive number π yields the equivalent inequalities $(0.65)^2\pi \le \pi r^2 \le 4\pi$. Since the area of a circle of radius r is $A = \pi r^2$, the smallest interval containing the areas A of all modern Mexican coins is $(0.65)^2\pi \le A \le 4\pi$. □

† Whenever you are asked in this text to generate a graph, generate it on a graphing calculator or computer.

(1) Data from *A Guide Book of Mexican Coins* by T. Buttrey, Jr. and C. Hubbard, Racine, Wisconsin: Western Publishing Company, Inc., 1969, pp. 173 and 208.

Example 6 Find the solution set of the inequality $1/x \le 1$.

SOLUTION The number $x = 0$ cannot be a solution of the inequality because x cannot be zero in the formula $1/x$. We treat the cases of positive and negative x separately. For positive x, multiplying both sides of $1/x \le 1$ by the positive number x gives the equivalent inequality $1 \le x$, which we can rewrite as $x \ge 1$. For negative x, we have to reverse the direction of the inequality when we multiply both sides by x, so we obtain $1 \ge x$. This condition automatically holds for all $x < 0$. Therefore, the original inequality, $1/x \le 1$ holds for $x \ge 1$ and for $x < 0$ and the solution set is $\{x : x < 0 \text{ or } x \ge 1\} = (-\infty, 0) \cup [1, \infty)$. This can be seen from the curve $y = 1/x$ and the line $y = 1$ in Figure 10. □

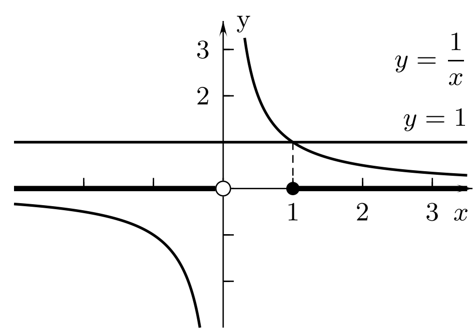


FIGURE 10

The following principles can often be used to deal with absolute values in inequalities.

$$
\begin{aligned}
&|f(x)| < k \text{ if and only if } -k < f(x) < k\\
&|f(x)| \le k \text{ if and only if } -k \le f(x) \le k\\
&|f(x)| > k \text{ if and only if } f(x) < -k \text{ or } f(x) > k\\
&|f(x)| \ge k \text{ if and only if } f(x) \le -k \text{ or } f(x) \ge k.
\end{aligned}
\qquad \textbf{(3)}
$$

Example 7 Find the solution set of the inequality $x^2 < 16$.

SOLUTION We can take square roots of both sides of $x^2 < 16$ because both numbers are nonnegative. This gives the equivalent inequality $\sqrt{x^2} < 4$, which can be written $|x| < 4$ or, eqivalently, $-4 < x < 4$. The solution set is the interval $(-4, 4)$. □

Question 4 Find the solution set of the inequality $x^2 \ge 16$.

Example 8 Solve $|1 - \frac{1}{2}x| > 2$ for x.

SOLUTION ONE SOLUTION: The condition $|1-\frac{1}{2}x| > 2$ holds if and only if $1-\frac{1}{2}x > 2$ or $1-\frac{1}{2}x < -2$. Subtracting 1 from both sides of these inequalities gives the equivalent condition, $-\frac{1}{2}x > 1$ or $-\frac{1}{2}x < -3$. Then multiplying both sides of both inequalities by the negative number -2 and reversing the direction of the inequality shows that x is a solution if $x < -2$ or $x > 6$. The solution set is $\{x : x < -2 \text{ or } x > 6\} = (-\infty, -2) \cup (6, \infty)$, as can be seen in Figure 11.

ANOTHER SOLUTION: Multiplying both sides of $|1 - \frac{1}{2}x| > 2$ by 2 gives the equivalent inequality $|2 - x| > 4$, which is satisfied if and only if the distance between 2 and x is greater than 4. Again we find that x is a solution if $x < -2$ or $x > 6$. □

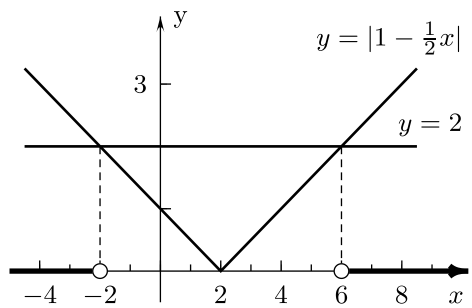


FIGURE 11

Responses 0.2

Response 1 For $x = 1$, $|2x-8| = |2(1)-8| = |-6| = 6$ • For $x = 7$, $|2x-8| = |2(7)-8| = |6| = 6$

Response 2 Figure R2 • The line $y = 6 + 2x$ is below or touches the line $y = 10$ for $x \le 2$, so that $6 + 2x \le 10$ for $x \le 2$.

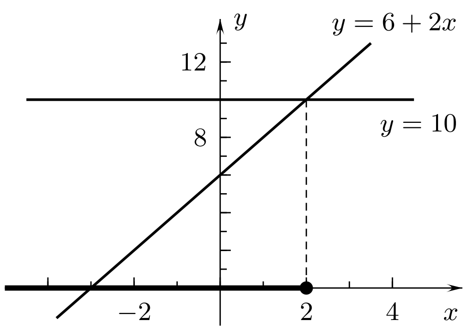


Figure R2

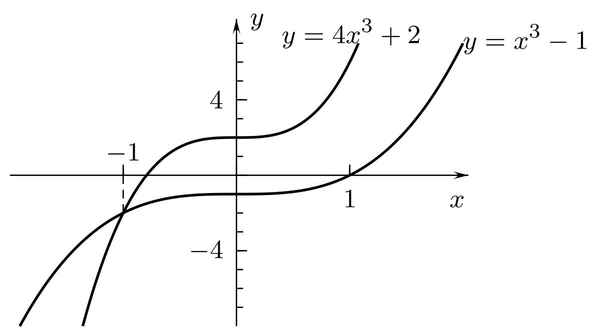


Figure R3

Response 3 Figure R3 • The curve $y = x^3 - 1$ intersects $y = 4x^3 + 2$ at $x = -1$ and is below it for $x > -1$, so that $x^3 - 1 < 4x^3 + 2$ for $x > -1$.

Response 4 $x^2 \ge 16$ • $\sqrt{x^2} \ge \sqrt{16}$ • $|x| \ge 4$ • $x \le -4$ or $x \ge 4$ • The solution set is $\{x : x \le -4 \text{ or } x \ge 4\} = (-\infty, -4] \cup [4, \infty)$.

Interactive Examples 0.2

Interactive solutions are on the web page http//www.math.ucsd.edu/˜ashenk/.[†]

1. Find the solution set of $x + 1 > 5x - 3$ and sketch it on an x-axis.

2. Find the widths w (inches) of cubes whose volumes $V = w^3$ (cubic inches) are in the interval $1 \le V \le 8$.

3 **(a)** Solve the equation $x^2 = 4|x|$ for x. **(b)** Generate $y = x^2$ and $y = 4|x|$ in the window $-6 \le x \le 6, -5 \le y \le 25$ on a calculator or computer to check your work.

4. Find the solution set of the inequalities $2 < 5 - 3x \le 4$.

5. Solve the inequality $\dfrac{x+3}{x} < 2$ for x. Generate the curve $\dfrac{x+3}{x}$ with the line $y = 2$ on a calculator or computer to check your result.

[†] In the published text the interactive solutions of these examples will be on an accompanying CD disk which can be run by any computer browser without using an internet connection.

Exercises 0.2

[A]Answer provided. [O]Outline of solution provided. [C]Graphing calculator or computer required.

CONCEPTS:

1. What happens to the valid inequalities **(a)** $-1 < 2$ and **(b)** $-1 \le 2$ when both sides are multiplied by 0?

2. What happens to the invalid inequalities **(a)** $5 < 4$ and **(b)** $5 \le 4$ when both sides are multiplied by 0?

3. What happens to the valid inequality $-3 < 2$ **(a)** when both sides are squared and **(b)** when square roots are taken of both sides?

BASICS:

1.[O] Give the solution set of the equation $|2x - 4| = 3$.

In Exercises 2 through 6 solve the inequalities and sketch the solution sets on x-axes.

2.[O] $x - 3 \ge 7 - 6x$

3.[O] $|3 - x| \ge 15$

4.[O] $3x^3 > -6$

5.[O] $|5x - 4| < 2$

6.[O] $-x^2 \le -4$.

7.[A] According to COULOMB'S LAW, the force of attraction between a negative charge of 1 statcoulomb and a positive charge of 1 statcoulomb that are r centimeters apart is $F = r^{-2}$ dynes. For what values of r is $F \le 4$?

8.[A] If 100 grams of salt are dissolved in a saline solution with a volume of V liters, then the concentration of the solution is $C = 100/V$ grams per liter. For what volumes is the concentration ≤ 50 grams per liter?

9. What are the widths w of squares whose perimeters P are < 16?

In Exercises 10 through 16 **(a)** solve the required equations and inequalities. [C]**(b)** Then generate the graphs of the indicated functions together in the given windows to check your conclusions.

10.[O] Find the intersection of the lines $y = x + 1$ and $y = 5 - 2x$. (Generate $y = x + 1$ and $y = 5 - 2x$ in $-2 \le x \le 3, -1 \le y \le 6$.)

11.[A] Find the solutions x of $|3x + 6| = 12$. (Generate $y = |3x + 4|$ and $y = 12$ in $-10 \le x \le 5, -5 \le y \le 18$.)

12.[O] Find the intersections of $y = x^4$ and $y = x^2$. (Generate $y = x^4$ and $y = x^2$ in $-1.5 \le x \le 1.5$, $-0.5 \le y \le 1.5$.)

13. Where do $y = x^4$ and $y = 8x$ intersect? (Generate $y = x^4$ and $y = 8x$ in $-3 \le x \le 3$, $-5 \le y \le 25$.)

14.[O] Find the solution set of $-5 < 3x + 4 < 7$. (Generate $y = -5, y = 3x + 4$, and $y = 7$ in $-5 \le x \le 2, -10 \le y \le 15$.)

15. What is the solution set of $x + 3 < 6x + 10$? (Generate $y = x + 3$ and $y = 6x + 10$ in $-4 \le x \le 2, -3 \le y \le 6$.)

16.[O] Find the solution set of $|6 - 2x| \ge 20$.

17. Find the solution set of $|5x - 4| > 2$.

EXPLORATION:

In Exercises 18 through 20, sketch the given sets on x-coordinate lines and then express the sets as intervals or unions of intervals.

18.[A] $\{x : -3 < x \leq 1 \text{ and } -2 \leq x < 3\}$

19.[A] $\{x : x \leq -3 \text{ and } x > 0\}$

20. $\{x : -3 < x \leq 1 \text{ or } -2 \leq x < 3\}$

In Exercises 21 through 34 **(a)** solve the equations or inequalities for x.[C] **(b)** In Exercises 21 through 24 generate graphs of the relevant functions in suitable windows as a partial check of your answers.

21.[O] $x^4 - 2x^2 = 2$

23.[A] $|x^2 + x| = 1$

24. $|x^2 - 2x - 3| \leq 0$

25.[O] $4 < 2x < x + 4$

26.[A] $1 \leq 4 - 2x < 2 + 3x$

27. $16 - 2x^3 > 0$

28.[A] $\frac{1}{4}x^4 < 4$

29. $1 < x^{3/4} \leq 8$

30. $\sqrt[3]{x} \geq \frac{1}{2}\sqrt[3]{x} - 5$

31. $(x+3)^{1/3} > 2$

32. $\sqrt{2x-1} \geq 2$

33. $1 < (x+1)^3 \leq 8.$

(End of Section 0.2)