Section 0.3

(5/16/07)

Power and exponential functions

OVERVIEW: *As we will see in later chapters, many mathematical models use power functions $y = x^n$ and exponential functions $y = b^x$. The definitions and basic properties of these functions, which are studied in precalculus courses, are reviewed in this section. We will discuss applications of these functions in later chapters, beginning with Chapter 4, where we study their derivatives. We also describe here how formulas for functions can be modified to translate, reflect, expand, and contract their graphs, The section closes with notes on the history of analytic geometry.*

Topics:

- ***Power functions*** $\mathbf{y = x^n}$ ***and their graphs***
- ***Vertical and horizontal translation***
- ***Reflection, magnification, and contraction***
- ***Exponential functions*** $\mathbf{y = b^x}$ ***and their graphs***
- ***Laws of exponents***
- ***Historical notes***

Power functions

A POWER FUNCTION is a function of the form $y = x^n$, where x is the variable and n is a constant. If n is a positive integer, then x^n equals the product of n x's, as in the formula $x^3 = x \cdot x \cdot x$. If n is a positive fraction, p/q, then $x^n = x^{p/q}$ is the qth root of the pth power of x, which also equals the pth power of the qth root of x. In the case of $n = \frac{5}{3}$, for example, we have

$$x^n = x^{5/3} = \sqrt[3]{x^5} = \left[\sqrt[3]{x}\right]^5.$$

If n is a negative integer or fraction, so that $n = -m$ with m a positive integer or fraction, then x^n equals $1/x^m$, as in the formulas

$$x^{-3} = \frac{1}{x^3} = \frac{1}{x \cdot x \cdot x}$$

$$x^{-5/3} = \frac{1}{x^{5/3}} = \frac{1}{\sqrt[3]{x^5}} = \frac{1}{\left[\sqrt[3]{x}\right]^5}.$$

To have formulas and identities involving x^n apply with zero exponents, x^0 is defined to be 1 for all x.

If n is not an integer or a fraction, it is IRRATIONAL and has an infinite decimal expansion, which is used to define x^n for positive x. The irrational number $\sqrt{2}$, for example, has the decimal expansion $\sqrt{2} = 1.4142135623\ldots$, and if we want to define $10^{\sqrt{2}}$, we let $n_1 = 1.4$ be the number obtained by taking only one digit after the decimal point in the expansion of $\sqrt{2}$, let $n_2 = 1.41$ be the number obtained by taking two digits after the decimal point, and so forth. This gives us an infinite string of rational numbers $n_1, n_2, n_3, \ldots$ that approaches $\sqrt{2}$. We say that $\sqrt{2}$ is the LIMIT of the numbers $n_1, n_2, n_3, \ldots$.[†] The numbers $10^{n_1}, 10^{n_2}, 10^{n_3}, \ldots$ are defined because the exponents $n_1, n_2, n_3, \ldots$ are rational. And, just as the numbers $n_1, n_2, n_3, \ldots$ approach their limit $\sqrt{2}$, the numbers $10^{n_1}, 10^{n_2}, 10^{n_3}, \ldots$ approach their limit, which is defined to be $10^{\sqrt{2}}$. The first seven of the numbers 10^{n_j} are calculated in the next question.

[†] We will discuss limits in Chapter 1.

Question 1 The following table gives the first seven of numbers $n_1, n_2, n_3, \ldots$ that approach $\sqrt{2} = 1.4142135623\ldots$. Use a calculator or computer to complete the second row of include approximate decimal values of $10^{n_1}, 10^{n_2}, 10^{n_3}, \ldots, 10^{n_7}$.

j	1	2	3	4
n_j	1.4	1.41	1.414	1.4142
$10^{n_j} \doteq$	25.118864	25.703958	25.941794	25.953943

The same procedure is used to define x^n for any irrational n and positive x. We let n_j denote the rational number obtained by taking j digits after the decimal point in the decimal expansion of n. Then the numbers n_j approach n and the numbers x^{n_j} approach x^n. The expression x^n is defined unless it involves dividing by zero, taking an even root of a negative number, or taking an irrational power of a negative number. Consequently, $y = x^n$ is defined for all x with three exceptions: it is not defined at 0 if n is negative; it is not defined for negative x if n is a fraction with an even denominator, and it is not defined for negative x is n is irrational.

The graphs of $\mathbf{y = x^n}$

Figures 1 through 4 show four curves $y = x^n$ with odd integers n. The curve $y = x$ with $n = 1$ in Figure 1 is a line through the origin. The shapes of the other curves are analyzed in the following questions.

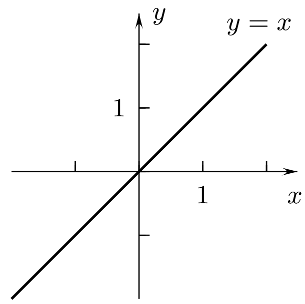


FIGURE 1

Question 2 **(a)** Why does $y = x^3$ in Figure 2 pass through the origin and why is it above the x-axis for $x > 0$ and below the x-axis for $x < 0$? **(b)** Use the formula $x^3 = x^2 \cdot x$ to explain why $y = x^3$ is much closer to the x-axis than $y = x$ for nonzero x very close to 0 and is much farther from the x-axis than $y = x$ for x very far from 0.

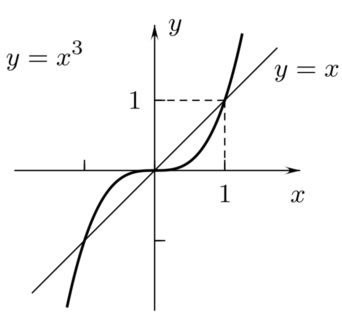


FIGURE 2

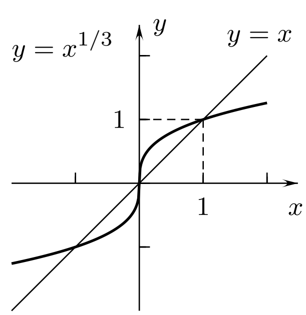


FIGURE 3

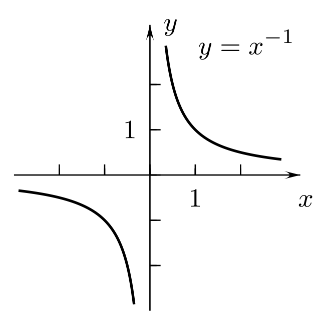


FIGURE 4

Notice that the curve $y = x^3$ in Figure 2 gets steeper as x moves away from 0.

Question 3 **(a)** Why does $y = x^{1/3}$ in Figure 3 pass through the origin and why is it above the x-axis for $x > 0$ and below the x-axis for $x < 0$? **(b)** Use the formula $x = x^{2/3} \cdot x^{1/3}$ to explain why $y = x$ is closer to the x-axis than $y = x^{1/3}$ for $-1 < x < 0$ and $0 < x < 1$ and is farther from the x-axis than $y = x^{1/3}$ for $x < -1$ and $x > 1$.

Notice that the curve $y = x^{1/3}$ in Figure 3 gets less steep as x moves away from 0.

Question 4 Explain why $y = x^{-1}$ in Figure 4 **(a)** does not intersect the y-axis and is above it for $x > 0$ and below it for $x < 0$, **(b)** is far from the x-axis for small positive and small negative x, and **(c)** is close to the x-axis for large positive and large negative x.

The reasoning in Questions 2 through 4 can be used to show that, for any constant n not equal to 0 or 1, the portion of $y = x^n$ for positive x is similar to the portion of one of the curves in Figures 2 through 4 for positive x. For $n > 1$ (Figure 5), the curve $y = x^n$, like $y = x^3$ in Figure 2, curves up to the right from the origin. For $0 < n < 1$ (Figure 6), the curve $y = x^n$, like $y = x^{1/3}$ in Figure 3, rises up from the origin but gets less steep as x moves to the right. For $n < 0$ (Figure 7), the curve $y = x^n$, like $y = x^{-1}$ in Figure 4, comes down to the right of the y-axis and approaches the y-axis as x moves to the right.

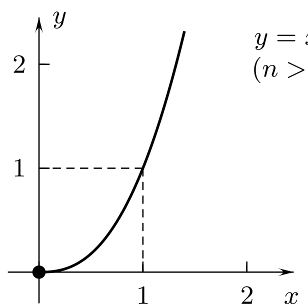


FIGURE 5

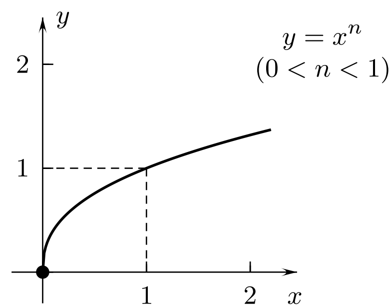


FIGURE 6

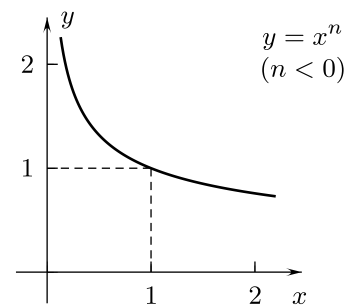


FIGURE 7

The nature of the graph $y = x^n$ for $x < 0$ depends on the value of n. There are three possibilities: $y = x^n$ is either an ODD function, is an EVEN function, or is not defined for negative x. We are using here the following definition.

Definition 1 *A function $y = f(x)$ is* EVEN *if $f(-x) = f(x)$ for all x in its domain and is* ODD *if $f(-x) = -f(x)$ for all x in its domain.*

The graphs of odd functions are symmetric about the origin; the graphs of even functions are symmetric about the y-axis.

The functions $y = x^3, y = x^{1/3}$, and $y = x^{-1}$ of Figures 2 through 4 are odd because $(-x)^3 = -x$ and $(-x)^{1/3} = -x^{1/3}$ for all x and $(-x)^{-1} = -x^{-1}$ for $x \neq 0$. Their graphs are symmetric about the origin.

The functions $y = x^2, y = x^{2/3}$, and $y = x^{-2}$ of Figures 8 through 10 are even because $(-x)^2 = x^2$ and $(-x)^{2/3} = x^{2/3}$ for all x and $(-x)^{-2} = x^{-2}$ for $x \neq 0$. Their graphs are symmetric about the y-axis.

The functions $y = x^{5/2}$, $y = x^{1/2}$, and $y = x^{-1/2}$ of Figures 11 through 12 are not defined for negative x because they involve the square root $x^{1/2} = \sqrt{x}$. Consequently, their graphs do not extend to the left of the y-axis.

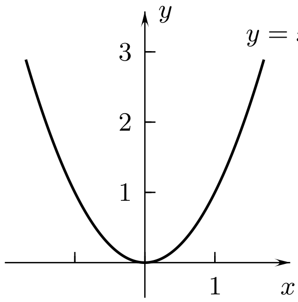


FIGURE 8

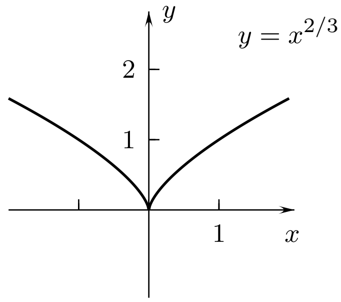


FIGURE 9

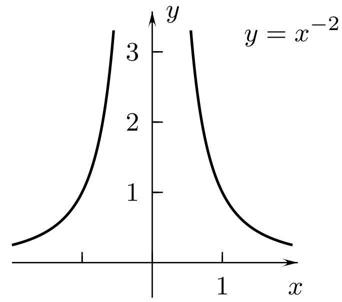


FIGURE 10

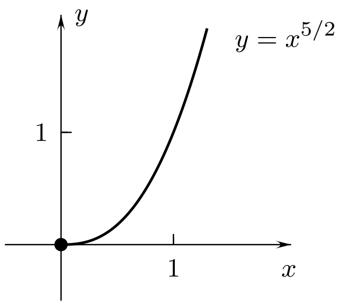


FIGURE 11

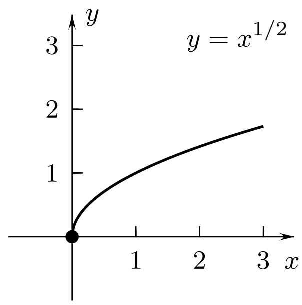


FIGURE 12

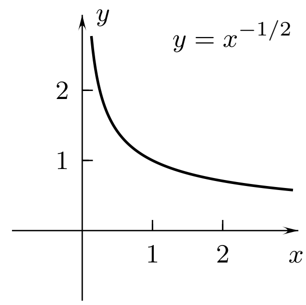


FIGURE 13

Example 1 Match the functions **(a)** $y = x^{-4}$, **(b)** $y = x^5$, and **(c)** $y = x^{1/4}$ to their graphs in Figures 14 through 16.

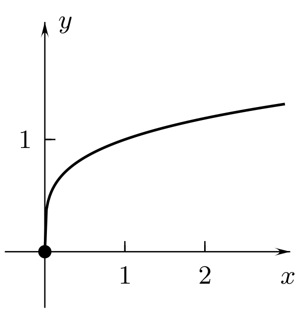


FIGURE 14

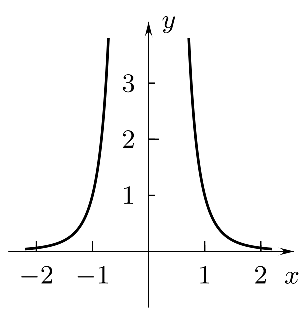


FIGURE 15

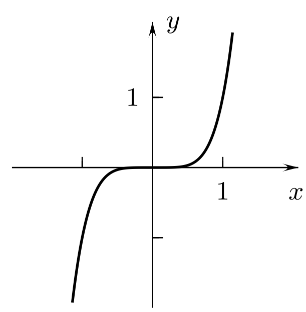


FIGURE 16

Solution **(a)** Here the power $n = -4$ is a negative number. Consequently, $y = x^{-4}$ is similar to $y = x^{-1}$ for positive x. The function $y = x^{-4}$ is even, so its graph is symmetric about the y-axis and is in Figure 15.
(b) The power 5 in this case is greater than 1, so $y = x^5$ is similar to $y = x^3$ for $x \geq 0$. The function $y = x^5$ is odd, so its graph is symmetric about the origin and is in Figure 16.
(c) Because $n = \frac{1}{4}$ is a fraction between 0 and 1, $y = x^{1/4}$ is similar to $y = x^{1/3}$ for $x \geq 0$, but since $n = \frac{1}{4}$ has an even denominator, the function is not defined for $x \leq 0$, and its graph is in Figure 14. □

Vertical and horizontal translation

If we add a positive constant to a function $y = f(x)$, we obtain the function $y = f(x) + k$, whose graph is obtained from the graph of f by raising it k units. Subtracting a positive constant k yields $y = f(x) - k$, whose graph is obtained by lowering the graph of $y = f(x)$ by k units (Figure 17). The raising or lowering of a graph is called VERTICAL TRANSLATION.

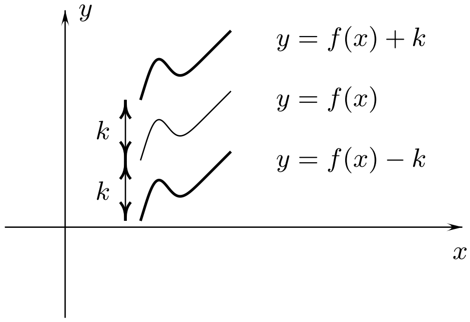


FIGURE 17

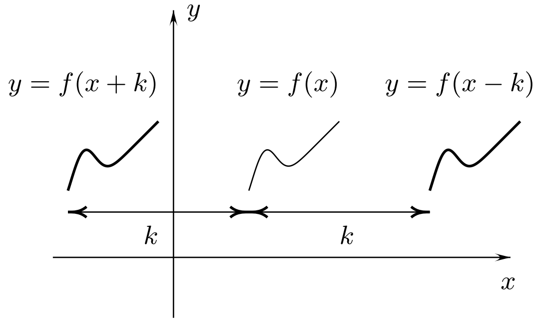


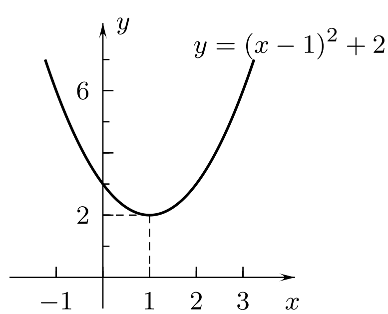
FIGURE 18

If, on the other hand, we add a positive constant k to the variable in the formula $y = f(x)$, we obtain $y = f(x + k)$, whose value at $x_0 - k$ is $f((x_0 - k) + k) = f(x_0)$, which is the value of $y = f(x)$ at x_0. Consequently, $y = f(x + k)$ is obtained by shifting the graph of f to the left k units, as shown in Figure 18. This action is called HORIZONTAL TRANSLATION. Subtracting a positive constant k from the variable gives $y = f(x - k)$, whose value at $x_0 + k$ is $f((x_0 + k) - k) = f(x_0)$, which is the value of $y = f(x)$ at x_0. Hence, $y = f(x - k)$ is the curve $y = f(x)$ shifted k units to the right, as is also shown in Figure 18.[†]

Example 2 Sketch the graph of the function $y = x^2 - 2x + 3$ by completing the square.

SOLUTION We complete the square in the formula $y = x^2 - 2x + 3$ by adding and subtracting the square of half the coefficient 2 of x. This gives $y = (x^2 - 2x + 1) - 1 + 3$ or $y = (x - 1)^2 + 2$. Its graph in Figure 19 is the curve $y = x^2$ translated up 2 units and 1 unit to the right. □

FIGURE 19



Reflection

Multiplying a function $y = f(x)$ by -1 gives the function $y = -f(x)$ whose value at x is the negative of the value of $y = f(x)$. Its graph is the mirror image of $y = f(x)$ relative to the x-axis. Multiplying the variable by -1 gives the function $y = f(-x)$ whose value at x is the value of $y = f(x)$ at $-x$. Its graph is the mirror image of $y = f(x)$ relative to the y-axis (Figure 20).

[†]Read this paragraph carefully: $y = f(x + k)$ is $y = f(x)$ shifted k units to the left, not to the right.

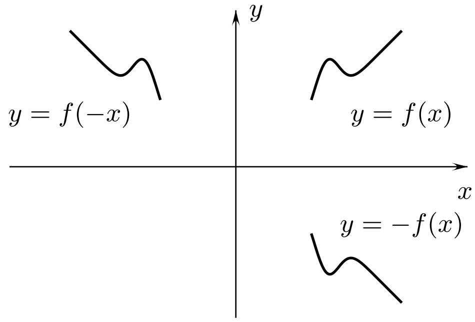


FIGURE 20

Question 5 Draw the graph of $y = -f(-x)$, where f is the function whose graph is in Figure 20.

Example 3 Sketch the graphs of **(a)** $y = -\sqrt{x}$ and **(b)** $y = \sqrt{-x}$.

SOLUTION **(a)** The graph of $y = -\sqrt{x}$ in Figure 21 is the mirror image relative to the x-axis of the curve $y = \sqrt{x}$ in Figure 12.
(b) The graph of $y = \sqrt{-x}$ in Figure 22 is the mirror image relative to the y-axis of $y = \sqrt{x}$. □

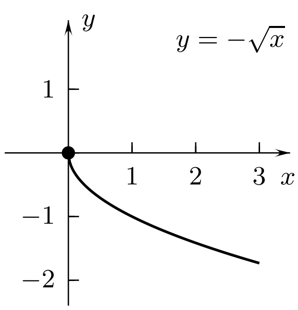


FIGURE 21

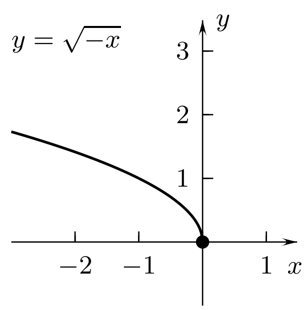


FIGURE 22

Magnification and contraction

If we multiply a function $y = f(x)$ by a constant $k > 1$, we obtain the function $y = kf(x)$, whose graph is obtained from $y = f(x)$ by multiplying the y-coordinate of every point on it by k. This magnifies the curve vertically, as shown by the middle and upper curves in Figure 23. Similarly, if we divide the function by $k > 1$, we obtain $y = \dfrac{1}{k}f(x)$, whose graph is obtained from $y = f(x)$ by dividing the y-coordinate of each point by k. This contracts the curve vertically, as shown by the lower curve in Figure 23.

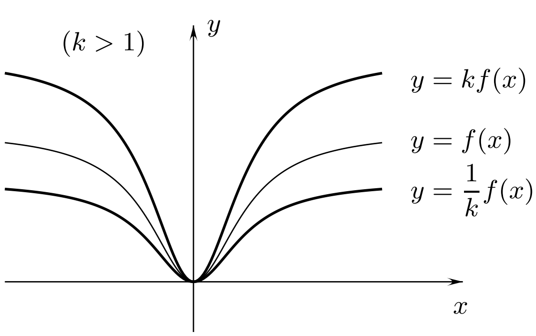


FIGURE 23

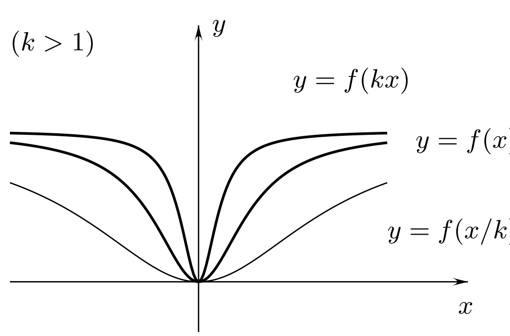


FIGURE 24

Multiplying the variable by a constant $k > 1$ contracts the graph horizontally since $y = f(kx)$ for positive k has the same value $f(k(x_0/k)) = f(x_0)$ at x_0/k as $y = f(x)$ has at x_0. Similarly, dividing the variable by $k > 1$ magnifies the graph horizontally (Figure 24).[†]

Example 4 The curve drawn with a heavy line in Figure 25 is the graph of $y = G(x)$. Is the other curve the graph of $y = \frac{1}{2}G(2x), y = \frac{1}{2}G(\frac{1}{2}x), y = \frac{1}{2}G(x)$, or $y = 2G(2x)$?

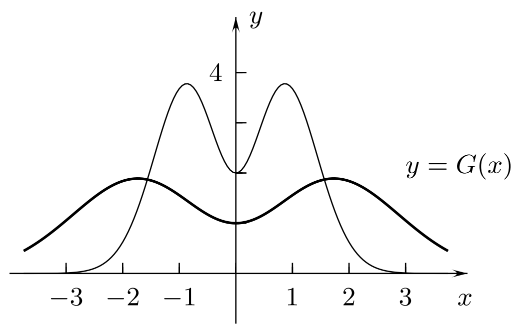


FIGURE 25

SOLUTION The second curve in Figure 25 has the equation $y = 2G(2x)$ because it is obtained by magnifying $y = G(x)$ by a factor of 2 vertically and contracting it by a factor of 2 horizontally. □

Exponential functions

An EXPONENTIAL FUNCTION is a function of the form $y = b^x$, where the exponent x is the variable and b is a positive constant, called the BASE. All exponential functions are defined and positive for all x, and their graphs pass through the point $(0, 1)$ since $b^0 = 1$ for any positive b.

If $b = 1$, then b^x is the constant function $y = 1$ (Figure 26). If b is greater than 1, then the graph approaches the x-axis on the left and curves up on the right (Figure 27). If $0 < b < 1$, then the graph approaches the x-axis on the right and curves up on the left (Figure 28).

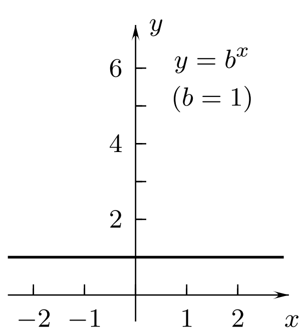


FIGURE 26

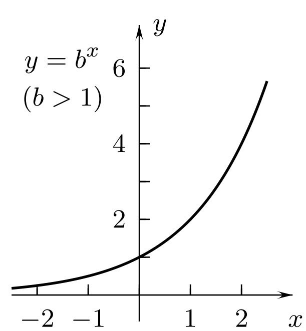


FIGURE 27

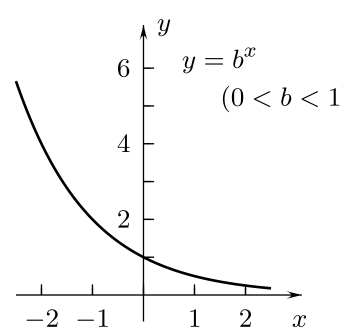


FIGURE 28

Notice that the function $y = b^x$ with positive $b \neq 1$ is neither even or odd; its graph is not symmetric about the y-axis nor about the origin.

[†]Read this paragraph carefully: for $k > 0$, $y = f(kx)$ is $y = f(x)$ contracted by a factor of k, not expanded by a factor of k.

Example 5 Draw the curve $y = 5 + 3(2^x)$.

SOLUTION The curve $y = 5 + 3(2^x)$ is $y = 2^x$ magnified vertically by a factor of 3 and then translated up 5 units. It is drawn in Figure 29, where the values $y(0) = 5 + 3(2^0) = 5 + 3 = 8$ and $y(2) = 5 + 3(2^2)$ $= 5 + 12 = 17$ on it have been plotted. The curve has $y = 5$ as a horizontal asymptote. □

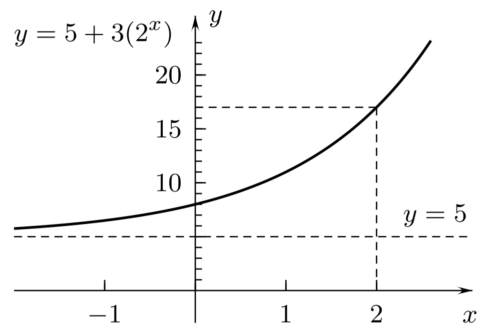


FIGURE 29

Question 6 Draw the curve $y = -5 - 3(2^{-x})$.

Example 6 A sample of radioactive radium-266 has mass $M(t) = 16\left(\frac{1}{2}\right)^{t/1620}$ grams at time t (years). **(a)** What is the mass of the sample at $t = 0, t = 1620$, and $t = 3240$ years and how are these numbers related? **(b)** Draw the graph of $M = M(t)$ in a tM-plane.

SOLUTION **(a)** The formula $M(t) = 16\left(\frac{1}{2}\right)^{t/1620}$ gives $M(0) = 16\left(\frac{1}{2}\right)^0 = 16$, $M(1620) = 16\left(\frac{1}{2}\right)^1 = 8$, and $M(3240) = 16\left(\frac{1}{2}\right)^2 = 4$. Consequently, $M(1620)$ is half of $M(0)$ and $M(3240)$ is half of $M(1620)$. (These calculations illustrate the fact that the HALF-LIFE of radium is 1620 years.)

(b) The graph of $M = M(t)$ in Figure 30 is obtained by plotting the values from part (a). □

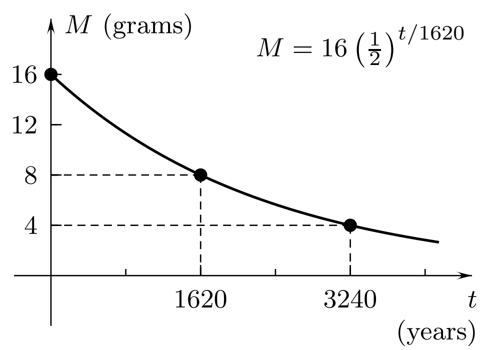


FIGURE 30

The natural exponential function

As we will see later, the most useful exponential function in calculus is the NATURAL EXPONENTIAL FUNCTION $y = e^x$, whose base is an irrational number $e = 2.7182818285\ldots$ that will be defined in Chapter 4. The graph of $y = e^x$ is shown in Figure 31.

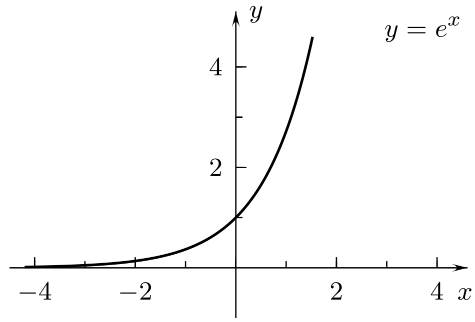


FIGURE 31

Laws of exponents

The following rules for working with exponents are valid for any numbers x and y if b is positive. If b is negative, they hold for all values of x and y such that all expressions involved are defined.

$$b^x b^y = b^{x+y} \tag{1}$$

$$(b^x)^y = b^{xy} \tag{2}$$

$$(bc)^x = b^x c^x \tag{3}$$

$$b^{-x} = \frac{1}{b^x} \tag{4}$$

The advantage of using exponential notation and these rules is illustrated in the next example.

Example 7 **(a)** Simplify the formula $y = \dfrac{\sqrt{b}}{\sqrt[3]{b}}$ without using fractional or negative exponents by taking the sixth power of both sides, simplifying, and taking the sixth root. **(b)** Simplify $y = \dfrac{\sqrt{b}}{\sqrt[3]{b}}$ by using fractional and negative exponents and rules **(1)** through **(4)**.

SOLUTION **(a)** Taking sixth powers of both sides of $y = \dfrac{\sqrt{b}}{\sqrt[3]{b}}$ gives

$$y^6 = \frac{\left(\sqrt{b}\right)^6}{\left(\sqrt[3]{b}\right)^6} = \frac{\left(\sqrt{b}\sqrt{b}\right)\left(\sqrt{b}\sqrt{b}\right)\left(\sqrt{b}\sqrt{b}\right)}{\left(\sqrt[3]{b}\sqrt[3]{b}\sqrt[3]{b}\right)\left(\sqrt[3]{b}\sqrt[3]{b}\sqrt[3]{b}\right)} = \frac{bbb}{bb} = b.$$

Then taking sixth roots yields $y = \sqrt[6]{b}$ since $y > 0$.

(b) Fractional exponents enable us to make a more direct calculation:

$$y = \frac{\sqrt{b}}{\sqrt[3]{b}} = b^{1/2}b^{-1/3} = b^{1/2-1/3} = b^{1/6} = \sqrt[6]{b}. \;\square$$

Historical notes

General algebraic equations were first studied in the sixteenth century, but without the modern convention of using a single letter for the unknown. The Italian physician and algebraist Gerolamo Cardano (1501–1572), for example, used the Latin sentence, *Cubus* $\overline{p}$ *6 rebus aequalis 20*, for the equation that we would write $x^3 + 6x = 20$. In Cardano's sentence, the word "cubus" denotes the cube of the unknown, $\overline{p}$ stands for "plus", "rebus" denotes the unknown, and "aequalis" means "equals."

The use of a single letter for the unknown and of exponents for positive integer powers was popularized by a treatise "La Géométrie," written in 1637 by the French philosopher Réné Descartes (1587–1650) as an appendix to a work on the philosophy of science.

Réné Descartes

(1587–1650)

Pierre Fermat

(1601–1665)

Descartes and a French lawyer Pierre Fermat (1601–1665) are considered the inventors of analytic geometry as a tool for giving geometric meaning to aspects of algebra and calculus. Greek mathematicians, including Euclid (ca. 300 BC), Archimedes (287–212 BC), and Apollonius (ca. 225 BC), used the equivalent of coordinate systems with the theory of proportions for studying geometric figures, and algebra was employed in the sixteenth century to solve geometric problems. It was Descartes and Fermat, however, who first studied curves defined by equations as well as by their geometric properties and who made extensive use of the association between the algebra of equations and the geometry of curves. Fermat's work was not published until after his death, more than forty years after the publication of Descartes' "La Géométrie," so Descartes often receives more credit for creating what now is known as analytic or "cartesian" geometry.

Fermat and Decartes generally used only positive coordinates. Negative coordinates were first used systematically by Isaac Newton (1642–1727) in his *Enumeration of Curves of Third Degree* (1676).

Fermat's name has been in the news in recent years because his "famous last theorem", that $x^n + y^n = z^n$ has no nonzero integer solutions x, y, and z for integers $n > 2$, has finally been proved.

Responses 0.3

Response 1 The table is completed below. (Notice that all of the values of 10^{n_j} begin with 25, the last five begin with 25.95, and the last two begin with 25.9545. This illustrates that the numbers approach the infinite decimal $25.9445\cdots = 10^{\sqrt{2}}$.)

j	1	2	3	4	5	6	7
x_j	1.4	1.41	1.414	1.4142	1.41421	1.414213	1.4142135
$10^{x_j} \doteq$	25.118864	25.703958	25.941794	25.953943	25.954341	25.954520	25.954550

Response 2 **(a)** $y = x^3$ passes through the origin because $0^3 = 0$, is above the x-axis for $x > 0$ because x^3 is positive for $x > 0$, and is below the x-axis for $x < 0$ because x^3 is negative for $x < 0$.
(b) $y = x^3$ is much closer to the x-axis than $y = x$ for very small nonzero x and is much farther from the x-axis than $y = x$ for large positive or negative x because $x^3 = x^2 \cdot x$ equals x multiplied by a very small positive number for small nonzero x and multiplied by a large positive number for large positive or negative x.

Response 3 **(a)** $y = x^{1/3}$ passes through the origin because $0^{1/3} = 0$ and is above the x-axis for positive x and below the x-axis for negative x because $x^{1/3}$ is positive for positive x and negative for negative x.
(b) $y = x$ is closer to the x-axis than $y = x^{1/3}$ for $-1 < x < 0$ and $0 < x < 1$ and is farther from the x-axis than $y = x^{1/3}$ for $x < -1$ and $x > 1$ because $x = x^{2/3} \cdot x^{1/3}$ equals $x^{1/3}$ multiplied by a positive number less than 1 if $-1 < x < 0$ or $0 < x < 1$ and equals $x^{1/3}$ multiplied by a number greater than 1 if $x < -1$ or $x > 1$.

Response 4 **(a)** $y = x^{-1}$ does not intersect the y-axis because $x^{-1} = 1/x$ is not defined at $x = 0$.
(b) $y = x^{-1}$ is far from the x-axis for very small x close to zero because $x^{-1} = 1/x$ is very large when x is very small.
(c) $y = x^{-1}$ is close to the x-axis if x is a large positive or negative number because then $x^{-1} = 1/x$ is very small.

Response 5 $y = -f(-x)$ is $y = f(x)$ reflected about the x-axis and about the y-axis. • Figure R5

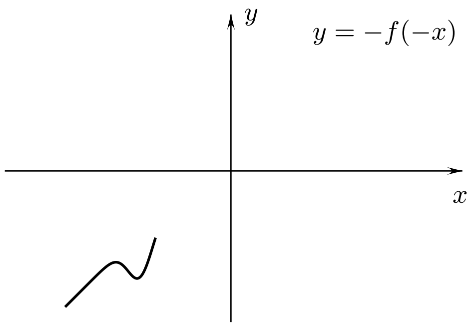


Figure R5

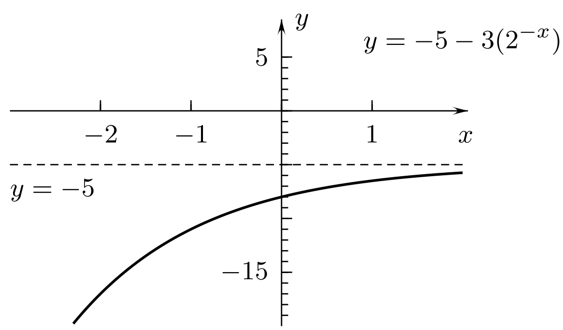


Figure R6

Response 6 $y = -5 - 3(2^{-x})$ is $y = 5 + 3(2^x)$ (Figure 29) reflected about the x- and y-axes. • Figure R6

Interactive Examples 0.3

Interactive solutions are on the web page http//www.math.ucsd.edu/˜ashenk/.†

1. Solve the equations **(a)** $x^{3/4} = 8$ and **(b)** $x^{4/3} = 16$ for x.

2. The curve in Figure 32 has the equation $y = a - \dfrac{1}{x-b}$ with constants a and b. What are those constants?

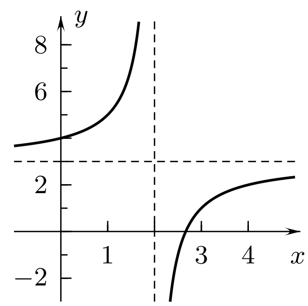


FIGURE 32

3. Do the two curves in Figure 33 have the equations $y = 5 + e^x, y = 5 - e^x, y = 5 + e^{-x}$, or $y = 5 - e^{-x}$?

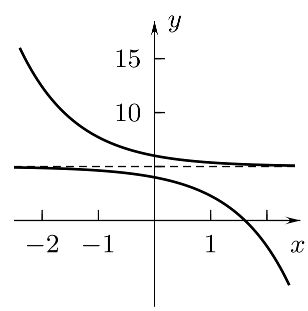


FIGURE 33

4. Solve the equation $\dfrac{9^{3x}}{9^x} = 3$ for x by using the fact that if $b^{x_1} = b^{x_2}$ with a positive $b \neq 1$, then $x_1 = x_2$.

5. **(a)** Determine the general shape of the curve $y = 3 - 1/x^2$ without generating it on a calculator or computer. **(b)** Sketch the curve by plotting at least one point on it.

6. **(a)** Determine the general shape of the curve $y = 1 + \sqrt{x-1}$ without generating it on a calculator or computer. **(b)** Sketch the curve by plotting at least one point on it.

† In the published text the interactive solutions of these examples will be on an accompanying CD disk which can be run by any computer browser without using an internet connection.

7. Figure 34 shows the graph of a function $y = P(x)$ and the curves $y = 2P(x)$ and $y = P(x) + 2$. Which curve is which?

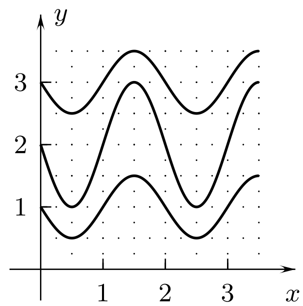


FIGURE 34

8. The curve drawn with a fine line in Figure 35 is the graph of $y = H(x)$. **(a)** Is the other curve above the x-axis the graph of $y = H(2x)$ or of $y = H(x/2)$? **(b)** Give equations in terms of H for the two curves below the x-axis.

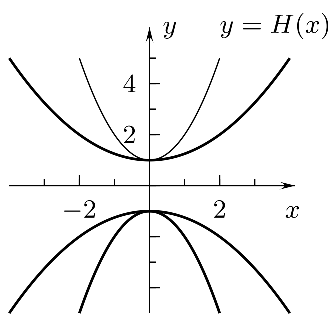


FIGURE 35

Exercises 0.3

^A^Answer provided. ^O^Outline of solution provided. ^C^Graphing calculator or computer required.

CONCEPTS:

C **1.** Generate the curves $y = x^2$ and $y = x^4$ together in the window $-1.5 \le x \le 1.5, -0.25 \le y \le 1.75$ and copy them on your paper. Then use the equation $x^4 = x^2(x^2)$ to explain why $y = x^4$ is below $y = x^2$ for some values of x and above it for others.

C **2.** Generate the curves $y = x^{-2}$ and $y = x^{-4}$ together in the window $-2.25 \le x \le 2.25, -0.5 \le y \le 2.5$ and copy them on your paper. Then use the equation $x^{-2} = x^2(x^{-4})$ to explain why $y = x^{-2}$ is below $y = x^{-4}$ for some values of x and above it for others.

3. Derive the identity $b^x b^y = b^{x+y}$ for a positive constant b in the case of $x = 2$ and $y = 3$ by writing $b^2 = b \cdot b$ and $b^3 = b \cdot b \cdot b$.

4. Derive the identity $(b^x)^y = b^{xy}$ for a positive constant b in the case of $x = 2$ and $y = 3$ by writing $(b^2)^3$ as $(b^2) \cdot (b^2) \cdot (b^2)$ and then writing $b \cdot b$ for b^2.

5. How can the function $z = x^y$ be either a power function or an exponential function?

BASICS:

6. Find all real solutions x of **(a)** $x^2 = k^2 + 4$, **(b)** $x^3 + 8 = k^3$, and **(c)** $\sqrt{x} = \sqrt{k} + 10$. Here k is a positive constant.

7. Solve the following equations for x by recognizing powers of 2 and 10 and using the fact that if $b^{x_1} = b^{x_2}$ with positive $b \neq 1$, then $x_1 = x_2$.

(a) $2^x = 8$ **(b)** $2^x = \frac{1}{8}$ **(c)** $10^x = 0.001$

(d) $10^x 10^x = 100$ **(e)** $(10^x)^x = 100$ **(f)** $\dfrac{2^{6x}}{2^x} = 16$

8. The middle curve in Figure 36 is $y = \dfrac{5}{1+x^2}$. The other curves are $y = \dfrac{5}{1+(2x)^2}$ and $y = \dfrac{5}{1+(x/2)^2}$. Which is which?

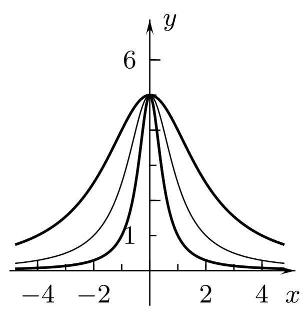


FIGURE 36

In Exercises 9 through 12 solve the equations for x.

9.[O] $x^3 \cdot x^4 = 5$

10.[A] $\dfrac{x^3}{x^5} = 16$

11. $\dfrac{x^{4/3}}{x} = 10$

12. $(x^4)^3 = 2$.

In Exercises 13 through 19 solve the equations for x by using the fact that if $b^{x_1} = b^{x_2}$ with positive $b \neq 1$, then $x_1 = x_2$.

13.[O] $2^x 3^x = 36$

14. $\dfrac{10^x}{5^x} = 8$

15.[A] $\left(\frac{1}{2}\right)^{2x} = 4^{-x}$.

16.[O] $7^{x^2} = (7^x)^2$

17.[A] $3^{x^2} = 9(3^x)$

18. $(2^x)^2 = 4^x$

19. $9^x = 27$

What are the domains of the functions in Exeercises 20 through 23?

20.[O] $y = x^{4/3}$

21.[A] $y = x^{1/4}$

22. $y = x^{-1}$

23. $y = x^{-3/2}$

Determine the general shapes of the curves in Exercises 24 through 31 without generating them on your calculator or computer by analyzing their equations. Then sketch them by plotting at least one point on each.

24.[A] $y = -5/x$

25. $y = -10 + 5/x^2$

26.[O] $y = 5 - 2/(x+1)^2$

27.[A] $y = 1 + \sqrt{x-1}$

28. $y = 4 + (x-1)^3$

29.[O] $y = 2^x - 2$

30.[A] $y = 10 - e^x$

31. $y = 10e^{-x/10}$

32. The lower curve in Figure 37 is $y = P(x)$ and the upper curve is $y = A + P(Bx)$ with constants A and B. What are A and B?

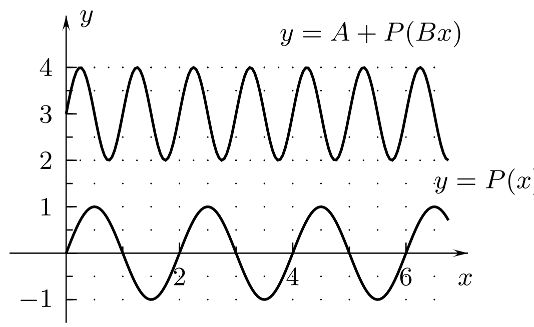


FIGURE 37

33. Figure 38 shows the curves $y = L(x+k)$ and $y = L(x-k)$ for a function $y = L(x)$ and a positive constant k. **(a)** Which is which and what is the value of k ? **(b)** Draw the graph of $y = L(x)$.

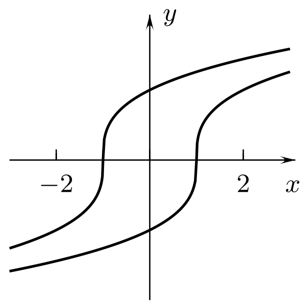


FIGURE 38

34.[A] According to Newton's law of gravity, an object that weighs one pound on the surface of the earth weighs $w = 16r^{-2}$ pounds when it is r thousand miles from the center of the earth. (The radius of the earth is 4 thousand miles.) Sketch the portion of $w = 16r^{-2}$ for $r \geq 4$ in an rw-plane.

35.[A] A boy throws a ball straight up in the air at time $t = -1$ (seconds) and catches it at $t = 1$. Because there is no air resistance, the ball is $h = 16 - 16t^2$ feet above his hand for $-1 \leq t \leq 1$. Sketch the portion of $h = 16 - 16t^2$ for $-1 \leq t \leq 1$ in a th-plane

36.[A] It costs a factory 5 dollars to manufacture each pint of a chemical, plus an overhead of 100 on each batch produced. The cost for a batch of x pints is therefore 100 + 5x dollars, and the average cost of a batch of x pints is $A = \dfrac{100+5x}{x} = \dfrac{100}{x} + 5$ dollars per pint. Draw the portion of $A = \dfrac{100}{x} + 5$ for $x > 0$ in an xA-plane.

37. A culture of bacteria contains 500 bacteria initiallly and, because the number doubles every 3 days, the culture contains $N = 500(2^{t/3})$ bacteria t days later. Draw the graph of this function in a tN-plane.

38. When air pressure is measured in ATMOSPHERES, the air pressure at the surface of the earth is 1 atmosphere. At an altitude $h < 80$ kilometers above the surface of the earth, the air pressure is $P = \left(\frac{1}{2}\right)^{h/5.8}$. Draw the graph of this function in an hP-plane.

EXPLORATION:

40.[O] $(5^{x^2})(5^x)^2 = 5^3$ for x.

41.[A] Solve $(5^{x^2})(5^x)^2 = 5$ for x.

42.[A] The curve $y = (2x)^2$ can be obtained from $y = x^2$ by contracting it horizontally or by magnifying it vertically. Explain.

43. The curve $y = e^{x+1}$ can be obtained from $y = e^x$ by horizonta1 translation or by vertical magnification. Explain.

44.[A] Which of the curves in Figures 39 through 44 is the graph of $y = a + b/x^2$ with $a > 0$? Is b positive or negative? Give your reasoning.

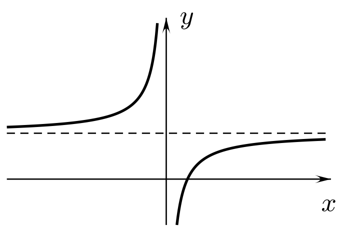


FIGURE 39

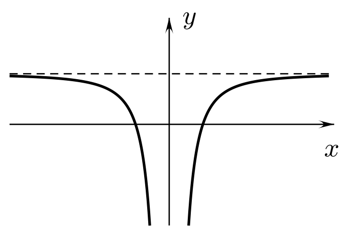


FIGURE 40

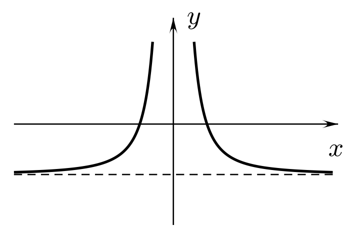


FIGURE 41

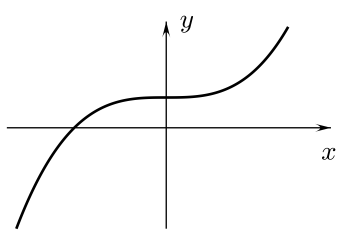


FIGURE 42

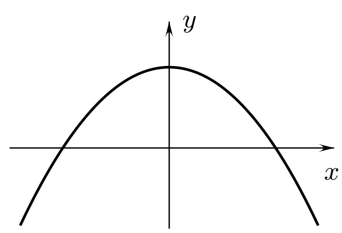


FIGURE 43

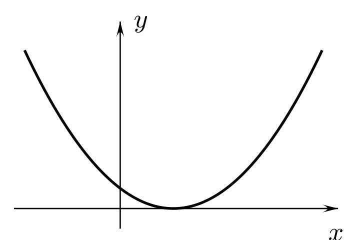


FIGURE 44

45. Which of the curves in Figures 39 through 44 is the graph of $y = a + bx^3$ for constants a and b? Is a positive or negative? Is b positive or negative? Give your reasoning.

46. Which of the curves in Figures 39 through 44 is the graph of $y = a + b/x$ for some constants a and b? Is a positive or negative? Is b positive or negative? Give your reasoning.

47. Which of the curves in Figure 45 has the equation $y = \dfrac{x^2 + a}{x^2 + b}$ **(a)** with $0 < a < b$, **(b)** with $0 < b < a$, and **(c)** with $a < 0$ and $b > 0$? Give your reasoning.

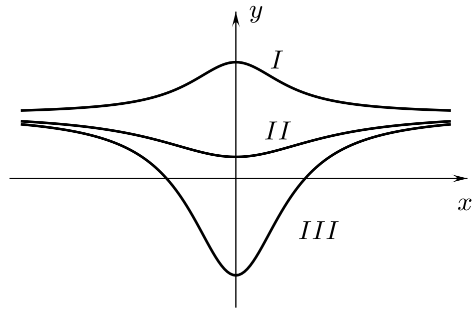


FIGURE 45

C **48.** Generate $y = x^2 + ax$ on your calculator or computer in the window $-4 \le x \le 4, -2 \le y \le 5$, first for $a = 0, 1$ and 2, and then for $a = 0, -1$, and -2. Explain how changing a changes the graph and why.

C **49.** How does changing b change the curve $y = 5x + bx^3$ and why? (Generate the curve for sample values of b in the window $-4 \le x \le 4, -30 \le y \le 30$ with y-scale $= 10$.)

50. The curve $y = M(x)$ is shown in Figure 46. Draw the curves **(a)** $y = M(x/2)$, **(b)** $y = M(2x)$, **(c)** $y = 2M(x)$, **(d)** $y = M(x)/2$, **(e)** $y = M(x)+2$, **(f)** $y = M(x-2)$, and **(g)** $y = 2M(x/2)$.

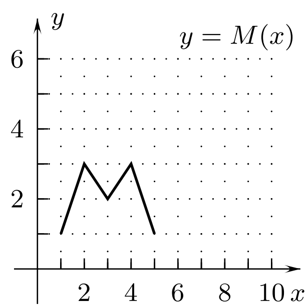


FIGURE 46

51. Figure 47 shows the graphs of $y = 1.5^x, y = e^x$, and $y = 6^x$. **(a)** Which is the upper curve, which is the middle curve, and which is the lower curve for $x > 0$? **(b)** Which is the upper curve, which is the middle curve, and which is the lower curve for $x < 0$?

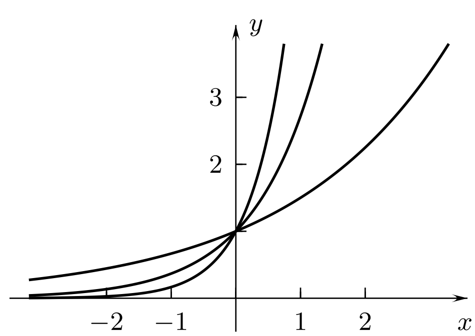


FIGURE 47

52. Find constants b and C such that $E(x) = Cb^x$ has the values in the following table:

x	0	1	2	3	4
$E(x)$	100	300	900	2700	8100

53. Figure 48 shows the curve $y = b^x$ with constant $b > 1$ and the mirror image of this curve about the origin. What is the equation of the second curve?

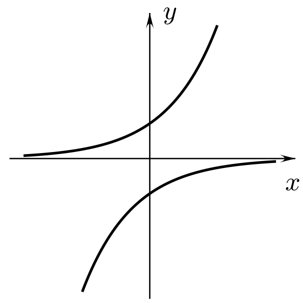


FIGURE 48

C **54.** Generate the curves $y = x^2$ and $y = 2^x$ in the window $-2.5 \le x \le 6, -2.5 \le y \le 25$ to see that the equation $x^2 = 2^x$ has two positive solutions and one negative solution. Find the positive solutions by trial and error and use a calculator or computer to find the approximate value of the negative solution.

55. Give a formula for the surface area $A = A(V)$ (square meters) of a cube as a function of its volume V (cubic meters) and draw the graph of this function.

(End of Section 0.3)