Section 0.5

(6/26/07)

Trigonometric and inverse trigonometric functions

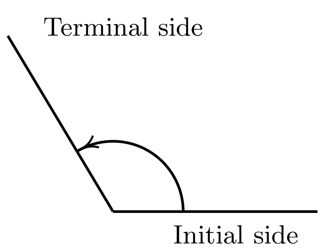
Overview: *If all you know about a triangle is its three angles, you cannot determine the lengths of its sides. You need to know how large the triangle is. You can, however, find the ratios of the lengths of the sides. These ratios for right triangles are given by the* trigonometric functions.*These functions are also used in analytic geometry and calculus to find ratios of distances from angles that are not in triangles. The* inverse trigonometric functions *are needed to find angles from ratios of distances. The definitions, basic properties, and graphs of these functions are reviewed in this section.*

Topics:

- ***Radian measure***
- ***Trigonometric functions and the unit circle***
- ***Some trigonometric identities***
- ***A direct definition of the tangent function***
- ***Graphs of the secant, cosecant, and cotangent functions***
- ***The inverse sine, cosine, tangent, and cotangent functions***
- ***The Law of Cosines and the Law of Sines***
- ***More trigonometric identities***

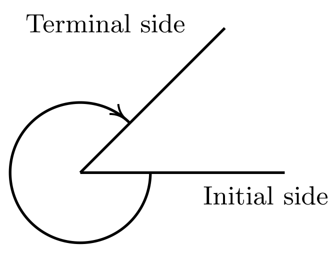
Radian measure

In calculus and analytic geometry an angle is determined by rotating one of its sides (the terminal side) from its other side (the initial side). The angle is positive if the rotation is counterclockwise (Figure 1) and negative if the rotation is clockwise (Figure 2). In calculus, angles are usually measured in radians. The radian measure of a positive angle with its vertex at the center of a circle of radius 1 (Figure 3) equals the length of the arc of the circle it subtends (marks off). The radian measure of a negative angle equals the negative of the length of the arc.



A positive angle

FIGURE 1



A negative angle

FIGURE 2

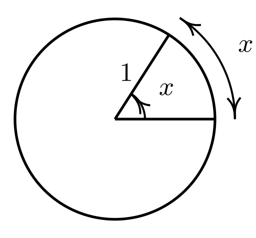


FIGURE 3

An angle x with $0 \le x \le 2\pi$ with its vertex at the center of a circle of radius r subtends an arc of length rx (Figure 4). Moreover, because the sector of the circle inside the angle x is the fraction $\dfrac{x}{2\pi}$ of the circle and the area of the circle is πr^2, the area of the sector is $\left(\dfrac{x}{2\pi}\right)(\pi r^2) = \frac{1}{2}xr^2$.

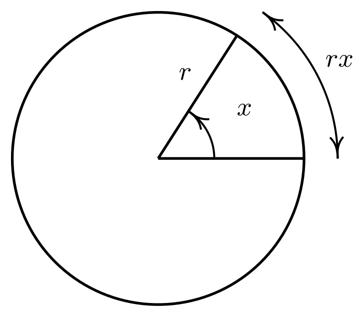


FIGURE 4

Arclength $= rx$

Area $= \frac{1}{2}xr^2$

Since the circumference of a circle of radius 1 is 2π, a half counterclockwise revolution is π radians, which corresponds to 180 degrees. Consequently, degrees can be converted to radians and radians to degrees with the formulas

$$\theta \text{ degrees} = [\theta \text{ degrees}]\left[\frac{\pi \text{ radians}}{180 \text{ degrees}}\right] = \left(\frac{\pi}{180}\right)\theta \;\text{ radians} \tag{1}$$

$$x \text{ radians} = [x \text{ radians}]\left[\frac{180 \text{ degrees}}{\pi \text{ radians}}\right] = \left(\frac{180}{\pi}\right)x \;\text{ degrees.} \tag{2}$$

The trigonometric functions

The trigonometric functions are the COSINE, SINE, TANGENT, SECANT, COSECANT, and COTANGENT and are denoted $y = \cos x, = \sin x, y = \tan x, y = \sec x, y = \csc x$, and $y = \cot x$. To define them, we place the vertex of an angle x at the origin of a uv-plane with equal scales on the coordinate axes and with the initial side of the angle along the positive u-axis, as in Figures 5 and 6. (We use uv-planes, rather than xy-planes, because we are using x as an angle.)

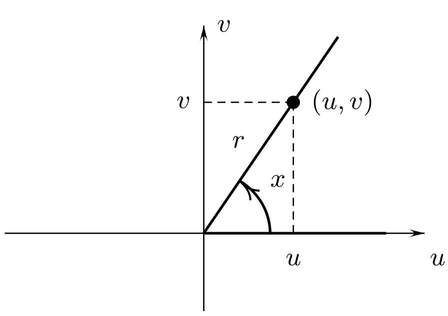


FIGURE 5

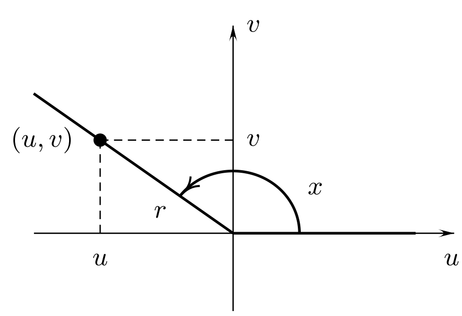


FIGURE 6

We pick any point (u, v), other than the origin, on the terminal side of the angle and calculate its distance $r = \sqrt{u^2 + v^2}$ to the origin. Then r is a positive number, but u and v may be positive, negative, or zero, depending on the value of x. In Figure 4, for instance, u and v are both positive, while u is negative and v is positive in Figure 5. The values of the trigonometric functions at the angle x are

$$\begin{aligned} \cos x &= \frac{u}{r}, & \sin x &= \frac{v}{r}, & \tan x &= \frac{v}{u} \\ \sec x &= \frac{r}{u}, & \csc x &= \frac{r}{v}, & \cot x &= \frac{u}{v}. \end{aligned} \tag{3}$$

These formulas do not depend on the choice of r because the right triangles, as in Figure 5 or Figure 6, formed by the axes and the terminal side of the same angle with two different values of r are similar and the ratios of the lengths of their corresponding sides are equal.

An angle x is called ACUTE if $0 < x < \frac{1}{2}\pi$. In this case, x is an angle in a right triangle (Figure 7), and definitions **(3)** take the form of the definitions from trigonometry:

$$\begin{aligned} \cos x &= \frac{\text{Adj}}{\text{Hyp}}, & \sin x &= \frac{\text{Opp}}{\text{Hyp}}, & \tan x &= \frac{\text{Opp}}{\text{Adj}}, \\ \sec x &= \frac{\text{Hyp}}{\text{Adj}}, & \csc x &= \frac{\text{Hyp}}{\text{Opp}}, & \cot x &= \frac{\text{Adj}}{\text{Opp}}. \end{aligned} \tag{4}$$

Here "Hyp" is the length r of the hypotenuse of the triangle, "Adj" is the length u of the side adjacent to the angle x, and "Opp" is the length v of the side opposite x.

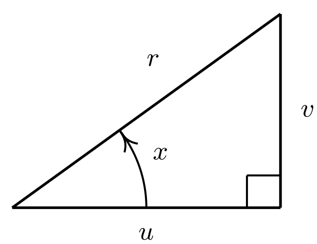


FIGURE 7

Question 1 Use Figure 7 to explain why $0 < \cos x < 1$ and $0 < \sin x < 1$ for positive acute angles x.

Example 1 Find the cosine, sine, tangent, secant, cosecant, and cotangent of the angle ψ (psi) in the right triangle of Figure 8.

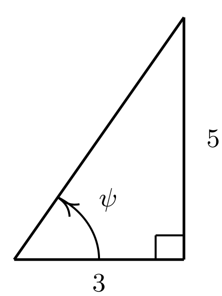


FIGURE 8

SOLUTION The length of the hypotenuse of the triangle is $\sqrt{3^2+5^2} = \sqrt{34}$, the length of the side opposite ψ is 5, and the length of the adjacent side is 3, so that

$$\cos\psi = \frac{3}{\sqrt{34}}, \quad \sin\psi = \frac{5}{\sqrt{34}}, \quad \tan\psi = \tfrac{5}{3}$$
$$\sec\psi = \frac{\sqrt{34}}{3}, \quad \csc\psi = \frac{\sqrt{34}}{5}, \quad \cot\psi = \tfrac{3}{5}. \square$$

The next example involves an angle x with $\frac{1}{2}\pi < x < \pi$. Such an angle is called OBTUSE.

Example 2 The point P in Figure 9 has coordinates $(20, 5)$, the point Q is 18 units from P, and the direction from P to Q is at an angle of 2.5 radians from the positive x-direction. What are the coordinates of Q?

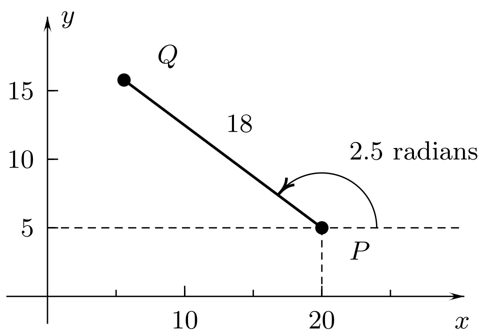


FIGURE 9

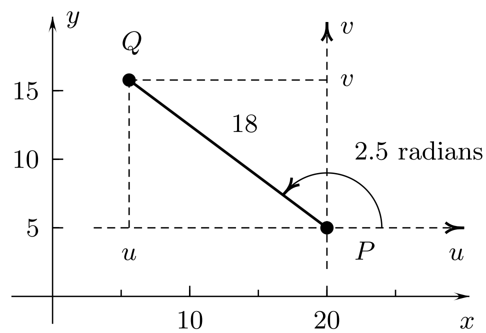


FIGURE 10

SOLUTION We introduce uv-axes with origin at P, as in Figure 10 and let (u, v) be the uv-coordinates of Q. Then by the first two definitions in **(3)** with $x = 2.5$ and $r = 18$, $\cos(2.5) = u/18$ and $\sin(2.5) = v/18$, so that $u = 18\cos(2.5)$ and $u = 18\sin(2.5)$. The x-coordinate of Q is the sum $x = 20 + u = 20 + 18\cos(2.5)$ of its u-coordinate and the x-coordinate of P. The y-coordinate of Q is the sum $y = 5 + v = 5 + 18\sin(2.5)$ of its v-coordinate and the y-coordinate of P. $\square$

C ***Question 2*** As a partial check of Example 2, find the approximate decimal values of the x- and y-coordinates of Q and compare the results with Figure 9.

The unit circle

We can simplify definitions **(3)** of the trigonometric functions by setting $r = 1$. To do this we consider points on the UNIT CIRCLE—the circle of radius 1 with its center at the origin—in a uv-plane (Figure 11). The point where the terminal side of the angle x intersects the unit circle is a distance $r = 1$ from the origin. Using its coordinates in **(3)** gives $\sin x = v/r = v$ and $\cos x = u/r = u$ so that $\sin x$ is the vertical v-coordinate of the point and $\cos x$ is its horizontal u-coordinate.

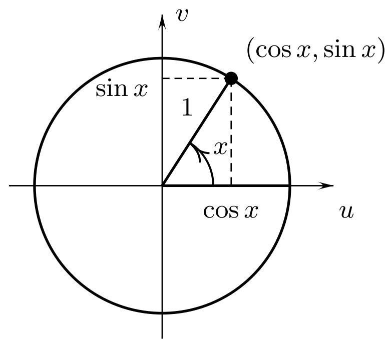


FIGURE 11

Figure 13 can be used to explain the shapes of the portions of the graphs $y = \sin x$ and $y = \cos x$ in Figures 12 and 13 for $0 \leq x \leq \frac{1}{2}\pi$. As the angle x in Figure 11 increases from 0 to $\frac{1}{2}\pi$, the intersection of its terminal side with the unit circle goes counterclockwise from the point $(1, 0)$ for $x = 0$ to the point $(0, 1)$ for $x = \frac{1}{2}\pi$. In this process the v-coordinate $\sin x$ increases from 0 to 1 and the u-coordinate $\cos x$ decreases from 1 to 0. Consequently, the graph of $y = \sin x$ in Figure 12 moves up from $y = 0$ at $x = 0$ to $y = 1$ at $x = \frac{1}{2}\pi$, while $y = \cos x$ in Figure 13 moves down from $y = 1$ at $x = 0$ to $y = 0$ at $x = \frac{1}{2}\pi$.

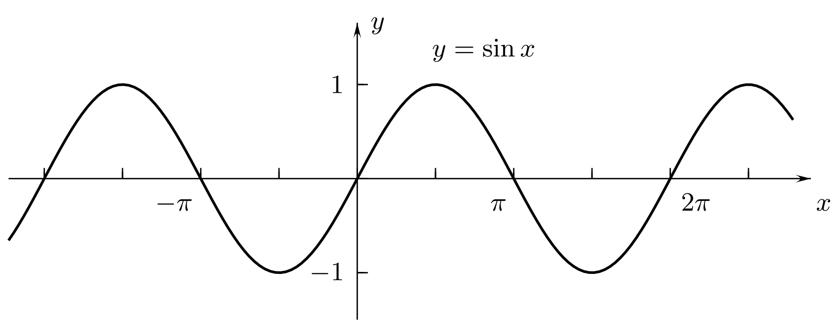


FIGURE 12

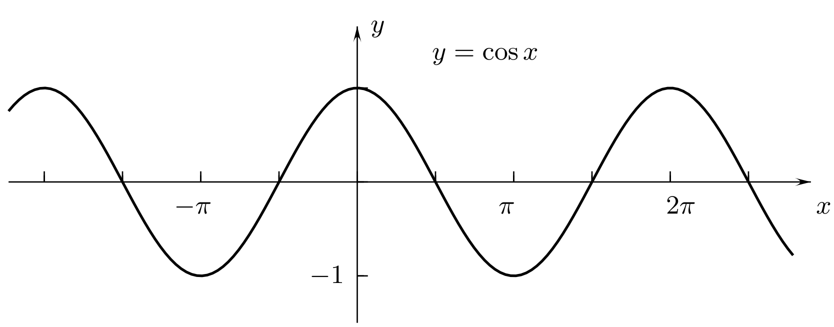


FIGURE 13

To understand the shapes of $y = \sin x$ and $y = \cos x$ for $\frac{1}{2}\pi \le x \le \pi$, we can use Figure 14. As x increases from $\frac{1}{2}\pi$ to π, the point on the unit circle moves from $(0, 1)$ at $x = \frac{1}{2}\pi$ to $(-1, 0)$ at $x = \pi$, so that $\sin x$ decreases from 1 to 0 and $\cos x$ decreases from 0 to -1 as shown in Figures 12 and 13. By visualizing the angle x in Figure 14 as it increases further, you can see that $y = \sin x$ and $y = \cos x$ have the overall shapes in Figures 12 and 13.

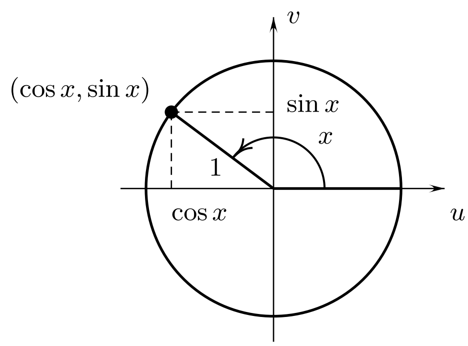


FIGURE 14

Some trigonometric identities

Since increasing the angle x by a full revolution gives an angle with the same terminal side,

$$\sin(x + 2\pi) = \sin x \quad \text{and} \quad \cos(x + 2\pi) = \cos x \quad \text{for all } x. \tag{5}$$

This property is described by saying that the sine and cosine functions are PERIODIC of PERIOD 2π.

The identity

$$\sin^2 x + \cos^2 x = 1 \tag{6}$$

holds for all x because the point $(\cos x, \sin x)$ is on the unit circle and its distance $\sqrt{\cos^2 x + \sin^2 x}$ to the origin is 1. Dividing both sides of **(6)** by $\cos^2 x$ or by $\sin^2 x$ yields, for all x such that the denominators are not zero,

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \quad \text{and} \quad \frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2}{\sin^2 x} = \frac{1}{\sin^2 x}.$$

Since $\tan x = \dfrac{\sin x}{\cos x}$, $\sec x = \dfrac{1}{\cos x}$, $\cot x = \dfrac{\cos x}{\sin x}$, and $\csc x = \dfrac{1}{\sin x}$, these equations can be written

$$\tan^2 x + 1 = \sec^2 x \tag{7}$$

$$\cot^2 x + 1 = \csc^2 x. \tag{8}$$

Formulas **(6)** through **(8)** are called PYTHAGOREAN IDENTITIES since they are based on the Pythagorean theorem.

As you can see in Figure 15, the terminal sides of angles x and $-x$ in a unit circle are symmetric about the u-axis, so that the points $(\cos x, \sin x)$ and $(\cos(-x), \sin(-x))$ are symmetric about the u-axis and

$$\cos(-x) = \cos x \quad \text{and} \quad \sin(-x) = -\sin x. \tag{9}$$

Similarly, the terminal sides of angles x and $x + \pi$ in Figure 16 are symmetric about the origin, so that

$$\cos(x + \pi) = -\cos x \quad \text{and} \quad \sin(x + \pi) = -\sin x. \tag{10}$$

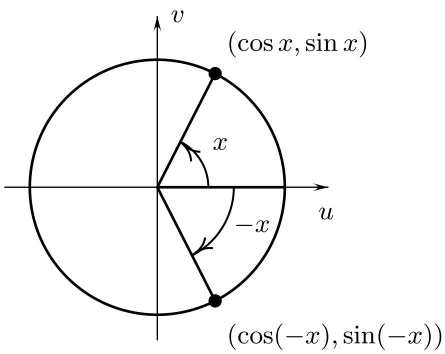


FIGURE 15

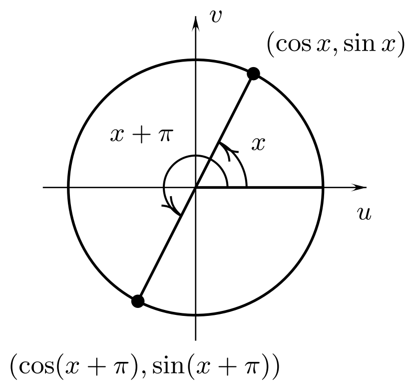


FIGURE 16

Identities **(9)** show that the cosine function is even and the sine function is odd, and identities **(9)** and **(10)** yield

$$\tan(-x) = \frac{\sin(-x)}{\cos(-x)} = \frac{-\sin x}{\cos x} = -\tan x \qquad \textbf{(11)}$$

$$\tan(x+\pi) = \frac{\sin(x+\pi)}{\cos(x+\pi)} = \frac{-\sin x}{-\cos x} = \tan x. \qquad \textbf{(12)}$$

This shows that the tangent function is odd and periodic of period π. Other trigonometric identities are listed at the end of this section.

A direct definition of the tangent function

We could define the tangent function in terms of the sine and cosine by the formula

$$\tan x = \frac{\sin x}{\cos x}$$

for all x such that $\cos x \neq 0$. All properties of the tangent function can then be determined from properties of the sine and cosine using this formula. It is useful, however, to have also a direct geometric definition of the tangent using the unit circle.

We draw the angle x and the vertical line $u = 1$ at the right of the unit circle. If the terminal side of the angle is to the right of the vertical axis, as in Figures 17 and 18, then the vertical, v-coordinate of its intersection with the line is $\tan x$ because the horizontal coordinate is $u = 1$ and by definition **(3)**, $\tan x = v/u = v$.†

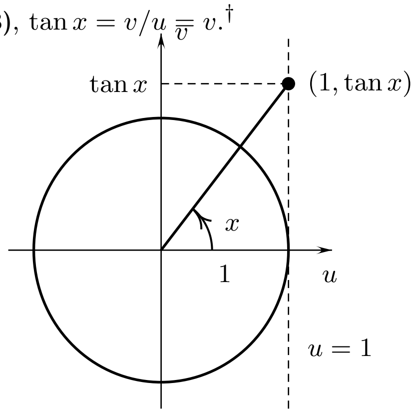


FIGURE 17

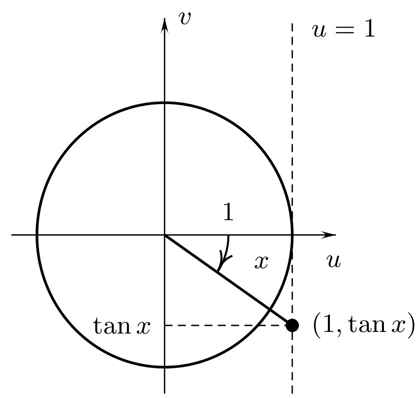


FIGURE 18

†The function $y = \tan x$ is called the "tangent" function because the line $u = 1$ in Figures 17 and 18 is tangent to the unit circle at $(1, 0)$.

If the terminal side of the angle is to the left of the vertical axis, as in Figures 19 and 20, we extend it backwards until it intersects the line $u = 1$ at the right of the unit circle and forms the terminal side of the angle $x - \pi$. The vertical, v-coordinate of the intersection is $\tan(x - \pi)$, which equals $\tan x$ because $y = \tan x$ is periodic of period π.

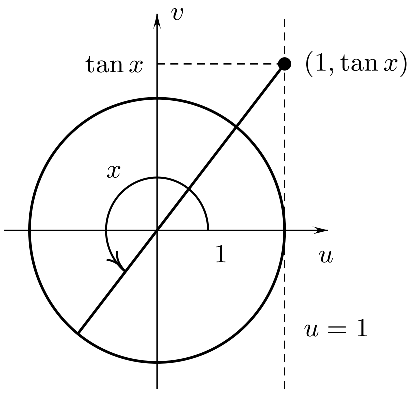


FIGURE 19

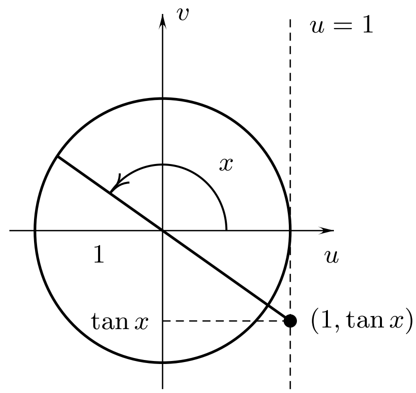
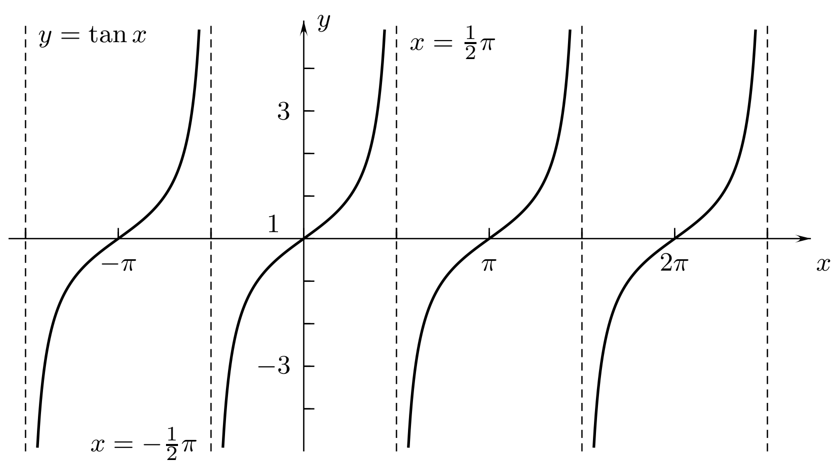


FIGURE 20

The shape of the graph of the tangent function in Figure 21 for $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$ can be seen from Figures 21 and 22. The curve $y = \tan x$ comes up to the right of its vertical asymptote $x = -\frac{1}{2}\pi$, passes through the origin, and then goes up to the left of its vertical asymptote $x = \frac{1}{2}\pi$. Its shape repeats every π units to the right and left because $y = \tan x$ has period π.

FIGURE 21



Example 3 Figure 22 shows a vertical pole 10 feet high and its shadow that is cast on the horizontal ground by the setting sun. Find a formula for the length $s = s(\theta)$ of the shadow as a function of the angle θ between the rays of the sun and the pole.

SOLUTION The length of the side opposite the positive acute angle θ in the right triangle of 24 is s and the length of the adjacent side is 10, so $\tan\theta = s/10$ and $s(\theta) = 10\tan\theta$. □

Question 3 Figure 23 shows the graph of $s = s(\theta)$ from Example 3. Use Figure 22 **(a)** to explain why $s(0) = 0$, **(b)** to explain why $s(\theta)$ is larger for larger θ, and **(c)** to find the value of $s\left(\frac{1}{4}\pi\right)$.

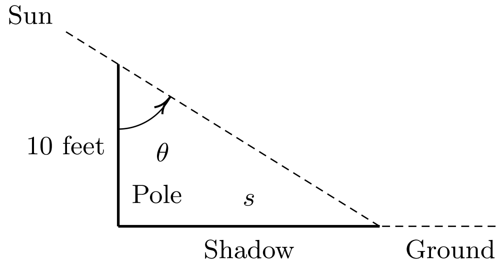


FIGURE 22

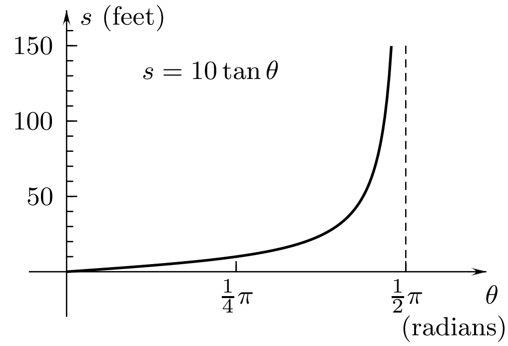
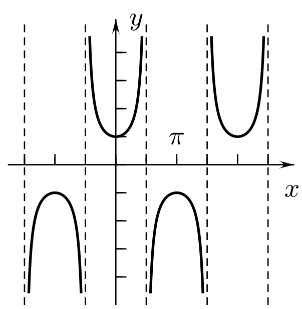


FIGURE 23

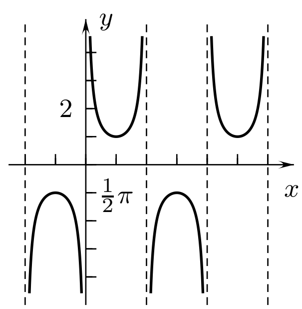
The graphs of the secant, cosecant, and cotangent functions

The graphs of $y = \sec x, y = \csc x$, and $y = \cot x$ are shown in Figures 24 through 26. Since $-1 \le \cos x \le 1$ for all x and $\sec x = 1/(\cos x)$, the values of $\sec x$ are ≥ 1 where $\cos x$ is positive, are ≤ -1 where $\cos x$ is negative, and the graph $y = \sec x$ has vertical asymptotes at the values of x where $\cos x = 0$. The shape of the graph of $\csc x = 1/(\sin x)$ can be determined similarly from properties of the sine function, and the shape of the curve $y = \cot x$ can be established with the formula $\cot x = (\cos x)/(\sin x)$.



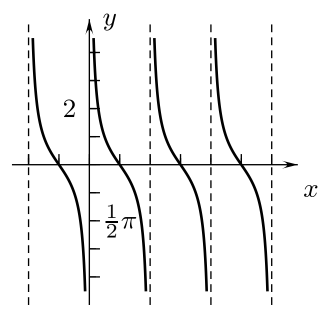
$y = \csc x$

FIGURE 24



$y = \sec x$

FIGURE 25



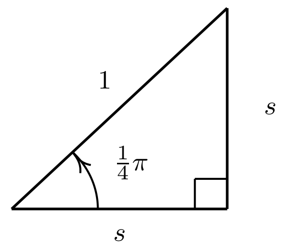
$y = \cot x$

FIGURE 26

Special angles

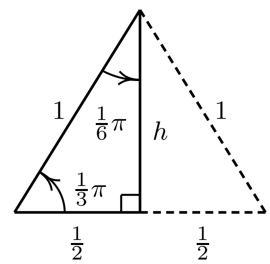
The values of the trigonometric functions at $x = \frac{1}{4}\pi$ can b e recalled by sketching an isosceles-right triangle with hypotenuse of length 1, as in Figure 27. The Pythagorean Theorem gives $s^2 + s^2 = 1^2$, so that $s^2 = \frac{1}{2}$, $s = 1/\sqrt{2}$, and consequently,

$$\sin\left(\tfrac{1}{4}\pi\right) = \frac{1}{\sqrt{2}}, \quad \cos\left(\tfrac{1}{4}\pi\right) = \frac{1}{\sqrt{2}}, \quad \tan\left(\tfrac{1}{4}\pi\right) = 1$$
$$\csc\left(\tfrac{1}{4}\pi\right) = \sqrt{2}, \quad \sec\left(\tfrac{1}{4}\pi\right) = \sqrt{2}, \quad \cot\left(\tfrac{1}{4}\pi\right) = 1.$$



$s = \sqrt{2}$

FIGURE 27



$h = \frac{1}{2}\sqrt{3}$

FIGURE 28

Similarly, the 30°-60°-right triangle in Figure 28 is half of an equilateral triangle with sides of length 1. Consequently, the length of its base is $\frac{1}{2}$, and, by the Pythagorean theorem, its height h satisfies $\left(\frac{1}{2}\right)^2 + h^2 = 1$, so that $h^2 = \frac{3}{4}$ and $h = \frac{1}{2}\sqrt{3}$. We obtain

$$\sin\left(\tfrac{1}{6}\pi\right) = \tfrac{1}{2}, \quad \cos\left(\tfrac{1}{6}\pi\right) = \tfrac{1}{2}\sqrt{3}, \quad \tan\left(\tfrac{1}{6}\pi\right) = \frac{1}{\sqrt{3}}$$
$$\sin\left(\tfrac{1}{3}\pi\right) = \tfrac{1}{2}\sqrt{3}, \quad \cos\left(\tfrac{1}{3}\pi\right) = \tfrac{1}{2}, \quad \tan\left(\tfrac{1}{3}\pi\right) = \sqrt{3}.$$

The cosecant, secant, and cotangent of these angles can be obtained from these values, and the values of the trigonometric functions at other integer multiples of $\frac{1}{6}\pi$ and $\frac{1}{4}\pi$ can be found by using a unit circle and the triangles in Figures 27 and 28 as needed.

The inverse sine, cosine, tangent, and cotangent

To find inverses of trigonometric functions, we have to restrict their domains since they are periodic and take each of their values at an infinite number of points. To obtain an INVERSE SINE FUNCTION, WE RESTRICT THE DOMAIN OF $y = \sin x$ TO $-\frac{1}{2}\pi \le x \le \frac{1}{2}\pi$. This gives the function whose graph is shown in Figure 29. Its inverse is denoted $y = \sin^{-1} x$ or $y = \arcsin x$. Because there are equal scales on the axes, the graph of the inverse sine in Figure 30 is obtained by reflecting the curve in Figure 29 about the line $y = x$.

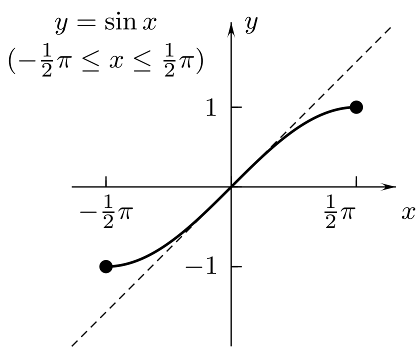


FIGURE 29

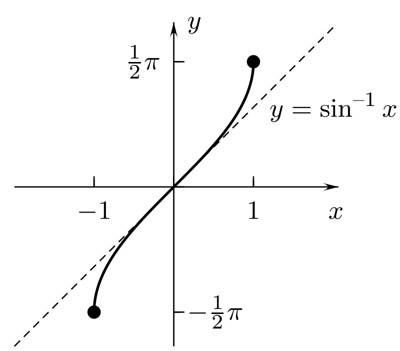


FIGURE 30

Since the domain of the function in Figure 29 is $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$ and its range is $[-1, 1]$, the domain of $y = \sin^{-1} x$ in Figure 30 is $[-1, 1]$ and its range is $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$. Therefore, $y = \sin^{-1} x$ is defined for $-1 \le x \le 1$ and $\sin^{-1} x$ is the angle y between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$ whose sine is x:

$$y = \sin^{-1} x \iff x = \sin y \quad \text{and} \quad -\tfrac{1}{2}\pi \le y \le \tfrac{1}{2}\pi. \tag{13}$$

To define an INVERSE COSINE, we restrict the domain of $y = \cos x$ to $0 \le x \le \pi$, as in Figure 31. The inverse is denoted $y = \cos^{-1} x$ or $y = \arccos x$ (Figure 32). It is defined for $-1 \le x \le 1$, and its value at x is the angle y between 0 and π whose cosine is x:

$$y = \cos^{-1} x \iff x = \cos y \quad \text{and} \quad 0 \le y \le \pi. \tag{14}$$

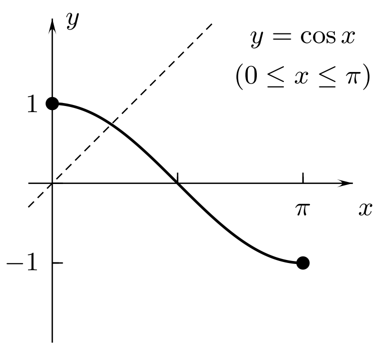


FIGURE 31

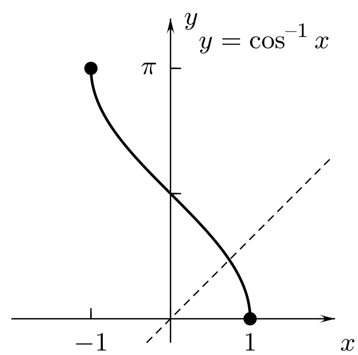


FIGURE 32

Question 4 **(a)** Why are $y = \sin^{-1} x$ and $y = \cos^{-1} x$ defined only for $-1 \le x \le 1$?

The inverse cosine can also be defined, for $-1 \le x \le 1$, by

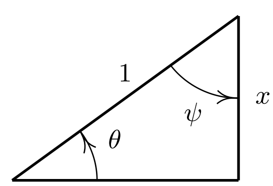
$$\cos^{-1} x = \tfrac{1}{2}\pi - \sin^{-1} x. \tag{15}$$

This formula is easy to remember because for positive x, as in Figure 33, $\sin^{-1} x$ and $\cos^{-1} x$ are the acute angles in a right triangle and, since the sum of the acute angles in a right triangle is $\frac{1}{2}\pi$ radians, $\sin^{-1} x + \cos^{-1} x = \frac{1}{2}\pi$.

$\theta = \sin^{-1} x$

$\psi = \cos^{-1} x$

FIGURE 33



To define the inverse tangent, we restrict the domain of $y = \tan x$ to $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$ (Figure 34). The inverse is denoted $y = \tan^{-1} x$ or $y = \arctan x$ (Figure 35). It is defined for all x, and its value at x is the angle y between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$ whose tangent is x:

$$y = \tan^{-1} x \iff x = \tan y \quad \text{and} \quad -\tfrac{1}{2}\pi < y < \tfrac{1}{2}\pi. \tag{16}$$

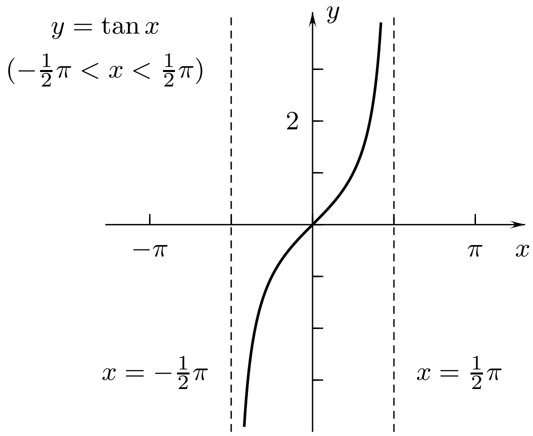


FIGURE 34

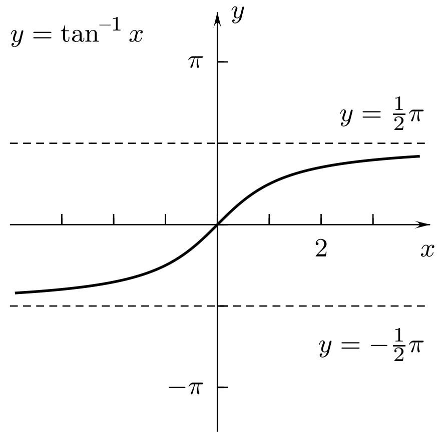
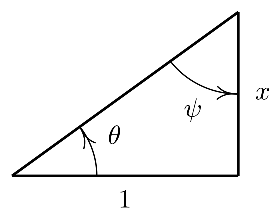


FIGURE 35

The inverse cotangent function can be defined by a formula analogous to **(15)** (see Figure 36):

$$\cot^{-1} x = \tfrac{1}{2}\pi - \tan^{-1} x. \qquad \textbf{(17)}$$



$\theta = \tan^{-1} x$
$\psi = \cot^{-1} x$
FIGURE 36

Example 4 Express the angle ψ in Figure 37 in terms of the lengths of the sides of the triangle by using **(a)** the inverse sine, **(b)** the inverse cosine, and **(c)** the inverse tangent.

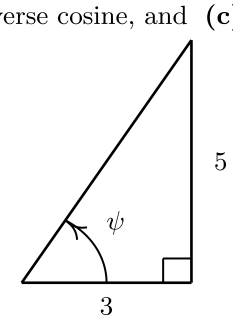


FIGURE 37

SOLUTION **(a)** The length of the hypotenuse of the right triangle in Figure 37 is $\sqrt{3^2+5^2} = \sqrt{34}$, so that $\sin\psi = 5/\sqrt{34}$. Then, since ψ is between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$, we have $\psi = \sin^{-1}(5/\sqrt{34})$ radians.
(b) We find $\cos\psi = 3/\sqrt{34}$, and since ψ is between 0 and π, $\psi = \cos^{-1}(3/\sqrt{34})$ radians.
(c) Because $\tan\psi = \frac{5}{3}$ and ψ is between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$, $\psi = \tan^{-1}(\frac{5}{3})$ radians. □

C ***Question 5*** Check that the formulas in the solution of Example 4 are consistent by calculating the approximate decimal value of ψ with all three formulas.

Example 5 A flashlight 2 meters above the ground is shining on a a vertical post 8 meters away, as shown in Figure 38. Give a formula for the angle θ between the beam of light and the horizontal as a function of the vertical distance h from the ground to the place where the light hits the post.

SOLUTION Because θ is the positive acute angle such that $\tan\theta = \dfrac{h-2}{8}$, it is given by $\theta = \tan^{-1}(\frac{1}{8}(h-2))$ radians. □

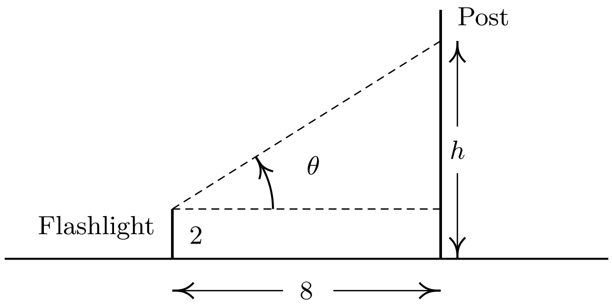


FIGURE 38

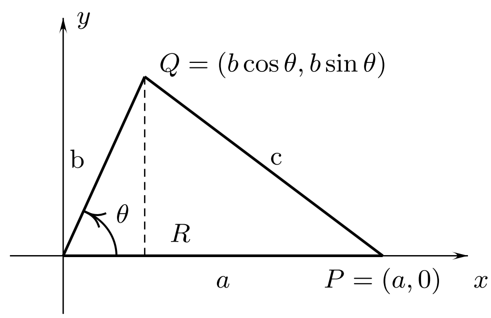


FIGURE 39

The law of cosines

Suppose you know an angle θ in a triangle and the lengths a and b of the adjacent sides, as in Figure 39. How can you find the length c of the side opposite θ? If θ is a right angle, then c is the length of the hypotenuse and a is the length of the side adjacent to θ, so that $\cos\theta = a/c$ and hence $c = a\sec\theta$. If the triangle is not a right triangle, we use instead the LAW OF COSINES, which can be derived as follows:

The points P, Q and R in Figure 39 form a right triangle and have coordinates $P = (a, 0)$, $Q = (b\cos\theta, b\sin\theta)$, and $R = (b\cos\theta, 0)$, so that by the Pythagorean theorem

$$\begin{aligned} c^2 &= \overline{PQ}^2 = \overline{RP}^2 + \overline{RQ}^2 \\ &= (a - b\cos\theta)^2 + (0 - b\sin\theta)^2 = a^2 - 2ab\cos\theta + b^2\cos^2\theta + b^2\sin^2\theta \\ &= a^2 + b^2(\cos^2\theta + \sin^2\theta) - 2ab\cos\theta. \end{aligned}$$

Then, since $\cos^2\theta + \sin^2\theta = 1$, we obtain $c^2 = a^2 + b^2 - 2ab\cos\theta$, and finally

$$c = \sqrt{a^2 + b^2 - 2ab\cos\theta}. \tag{18}$$

This is the law of cosines.

Example 6 Two sides of a triangle are 8 and 10 meters long and the angle between them is $\frac{4}{5}\pi$ radians. How long is the other side?

SOLUTION By **(18)** with $a = 8, b = 10$, and $\theta = \frac{4}{5}\pi$, the length of the third side is $c = \sqrt{8^2 + 10^2 - 2(8)(10)\cos(\frac{4}{5}\pi)} \doteq 17.13$ meters. □

Question 6 What is the law of cosines if the angle θ is a right angle?

The law of sines

The law of cosines enables us to use the lengths of two sides in a triangle and the enclosed angle to find the length of the third side. If we know instead two of the angles but the length of only one side, we can use the LAW OF SINES to find the lengths of the other sides. Suppose that α and β are two angles in a nonright triangle and that a and b are the lengths of the sides opposite them, as in Figure 40. We let h denote the altitude of the triangle as in the drawing. Then $h = b\sin\alpha$ and $h = a\sin\beta$ so that $b\sin\alpha = a\sin\beta$ and

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b}.$$

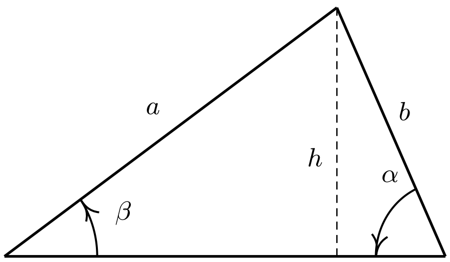


FIGURE 40

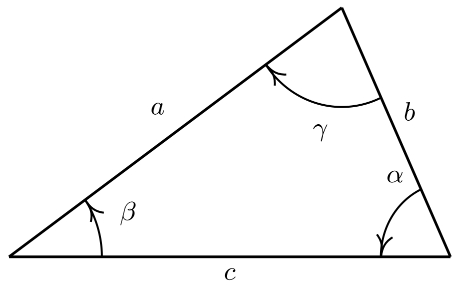


FIGURE 41

Applying this reasoning to all three angles in a triangle yields

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c} \tag{19}$$

where, as in Figure 41, c is the length of the side opposite the angle γ. This is law of sines.

Example 7 Two angles in a triangle are 0.3 and 0.5 radians. The side opposite the angle of 0.5 radians is 10 yards long. How long is the side opposite the angle of 0.3 radians? Give an approximate decimal answer.

SOLUTION With s the length of the side opposite the angle of 0.3 radians, $\dfrac{\sin(0.3)}{s} = \dfrac{\sin(0.5)}{10}$, so that $s = 10\left[\dfrac{\sin(0.3)}{\sin(0.5)}\right] \doteq 6.16$ yards. □

Question 7 What does the law of sines give if the angle α in Figure 41 is a right angle?

More trigonometric identities

Identities **(25)** through **(28)** below are derived for positive acute angles in Exercises 4 and 5. The other identities are derived in trigonometry classes.

$$\cos(\tfrac{1}{2}\pi - x) = \sin x \tag{20}$$

$$\sin(\tfrac{1}{2}\pi - x) = \cos x \tag{21}$$

$$\cos(x + \tfrac{1}{2}\pi) = -\sin x \tag{22}$$

$$\sin(x + \tfrac{1}{2}\pi) = \cos x \tag{23}$$

$$\sin(\alpha + \beta) = \cos\alpha\sin\beta + \sin\alpha\cos\beta \tag{24}$$

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{25}$$

$$\sin\alpha\sin\beta = \tfrac{1}{2}[\cos(\alpha - \beta) - \cos(\alpha + \beta)] \tag{26}$$

$$\cos\alpha\cos\beta = \tfrac{1}{2}[\cos(\alpha - \beta) + \cos(\alpha + \beta)] \tag{27}$$

$$\sin\alpha\cos\beta = \tfrac{1}{2}[\sin(\alpha - \beta) + \sin(\alpha + \beta)] \tag{28}$$

$$\sin(2\alpha) = 2\sin\alpha\cos x \tag{29}$$

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha \tag{30}$$

$$\sin^2\alpha = \tfrac{1}{2}\,[1 - \cos(2\alpha)] \tag{31}$$

$$\cos^2\alpha = \tfrac{1}{2}\,[1 + \cos(2\alpha)] \tag{32}$$

$$\sin\alpha - \sin\beta = 2\sin\left(\tfrac{1}{2}(\alpha - \beta)\right)\cos\left(\tfrac{1}{2}(\alpha + \beta)\right) \tag{33}$$

$$\cos\alpha - \cos\beta = 2\sin\left(\tfrac{1}{2}(\alpha - \beta)\right)\sin\left(\tfrac{1}{2}(\alpha + \beta)\right) \tag{34}$$

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}. \tag{35}$$

Responses 0.5

Response 1 $0 < \cos x < 1$ and $0 < \sin x < 1$ for $0 < x < \frac{1}{2}\pi$ since $\sin x$ and $\cos x$ equal the lengths of legs of a right triangle divided by the length of its hypotenuse and the legs are shorter than the hypotenuse.

Response 2 The x-coordinate of Q is $x = 20 + 18\cos(2.5) \doteq 5.58$ and its y-coordinate is $y = 5 + 18\sin(2.5) \doteq 15.77$, and these values are consistent with Figure 9.

Response 3 **(a)** $s(0) = 0$ because there is no shadow when the sun is directly over the pole.
(b) $s(\theta)$ is larger for larger θ because the shadow is longer when the sun is lower in the sky.
(c) $s(\frac{1}{4}\pi) = 10$ because with the sun at this angle the right triangle in Figure 21 is isosceles.

Response 4 $y = \sin^{-1} x$ and $y = \cos^{-1} x$ are only defined for $-1 \le x \le 1$ because they are angles whose sine and cosine are x, respectively, and all values of the sine and cosine are between -1 and 1.

Response 5 $\psi = \sin^{-1}(5/\sqrt{34}) \doteq 1.03038$ radians • $\psi = \cos^{-1}(3/\sqrt{34}) \doteq 1.03038$ radians • $\psi = \tan^{-1}(\frac{5}{3}) \doteq 1.03038$ radians

Response 6 If the angle θ in Figure 39 is a right angle, then $\cos\theta = 0$ and the law of cosines $c = \sqrt{a^2 + b^2 - 2ac\cos\theta}$ becomes the Pythagorean theorem $c = \sqrt{a^2 + b^2}$.

Response 7 If the angle α in Figure 41 is a right angle, as in Figure R7, then $\sin\alpha = 1$ and the law of sines $\dfrac{\sin\alpha}{a} = \dfrac{\sin\beta}{b} = \dfrac{\sin\gamma}{c}$ becomes $\dfrac{1}{a} = \dfrac{\sin\beta}{b} = \dfrac{\sin\gamma}{c}$. • This is equivalent to $\sin\beta = b/a$ and $\sin\gamma = c/a$, which can be obtained from the definitions of $\sin\beta$ and $\sin\gamma$.

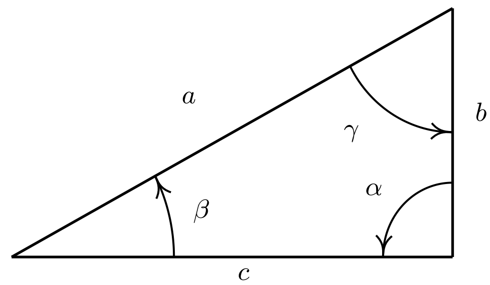


Figure R7

Interactive Examples 0.5

Interactive solutions are on the web page http//www.math.ucsd.edu/˜ashenk/.[†]

1. Give the radian measure of $-60°$. Then draw the angle in a unit circle and find its sine, cosine, and tangent.

2. Find $\cos\theta$, $\sin\theta$, and $\tan\theta$ for the angle θ in Figure 42.

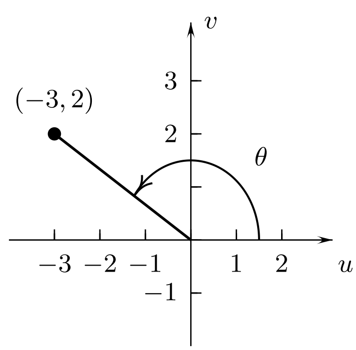


FIGURE 42

3. Use $\sqrt{2}$ and $\sqrt{3}$ as needed to give the exact value of $\csc(-\frac{3}{4}\pi)$.

[†]In the published text the interactive solutions of these examples will be on an accompanying CD disk which can be run by any computer browser without using an internet connection.

4. The direction from Helsinki, Finland toward St. Petersburg, Russia is 7.2° south of east, and Helsinki and St. Petersburg are 298 kilometers apart (Figure 43). How far is St. Petersburg south and how far is it east of Helsinki? Give exact and approximate decimal answers.

FIGURE 43

5. Use π as needed to give the exact value of $\tan^{-1}(-\sqrt{3})$.

6. One leg of a right triangle is 8 feet long and the hypotenuse is 10 feet long. What is the smallest angle in the triangle? Give the exact answer and its approximate decimal value.

Exercises 0.5

$^{\mathsf{A}}$Answer provided. $^{\mathsf{O}}$Outline of solution provided. $^{\mathsf{C}}$Graphing calculator or computer required.

CONCEPTS:

1. Figure 44 shows an angle of 1 radian with its center at a circle of radius 1 and an equilateral triangle with sides of length 1. Use this drawing to explain why 1 radian is a few degrees less than 60°.

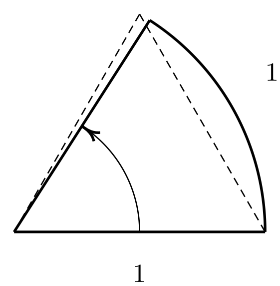


FIGURE 44

2. Draw two right triangles of different sizes that contain an angle α such that $\sin\alpha = 0.6$.

3. The two angles labeled β in Figure 45 equal because the vertical lines are parallel. Use Figure 45 to show that $\alpha + \beta = \frac{1}{2}\pi$.

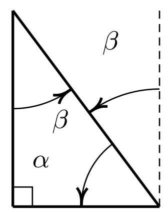


FIGURE 45

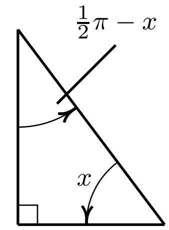


FIGURE 46

4. **(a)** Use the result of Exercise 3 to explain Figure 46. **(b)** Use Figure 46 to derive, for $0 < x < \frac{1}{2}\pi$, the identities, $\cos(\frac{1}{2}\pi - x) = \sin x$ and $\sin(\frac{1}{2}\pi - x) = \cos x$.

5. Use Figure 47 to derive, for $0 < x < \frac{1}{2}\pi$, the identities, $\cos(x + \frac{1}{2}\pi) = -\sin x$ and $\sin(x + \frac{1}{2}\pi) = \cos x$.

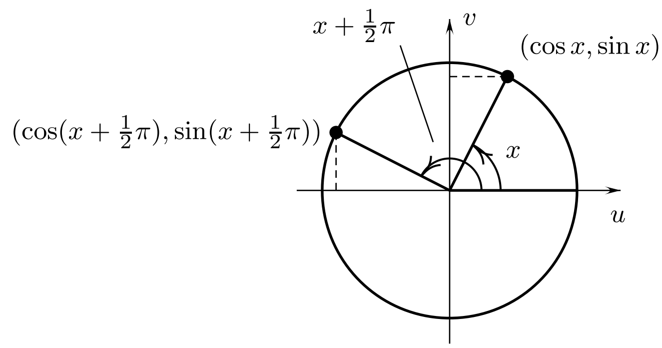


FIGURE 47

6. Six trigonometric functions are used to simplify formulas and display patterns in formulas. We could, however, do everything with only the sine function. For instance, the second formula $\cos x = \sin(\frac{1}{2}\pi - x)$ from Exercise 4 shows how the sine could be used in place of the cosine. Give similar formulas for $\tan x, \cot x, \sec x$, and $\csc x$ in terms of the sine.

BASICS:

7.$^{\text{O}}$ Give **(a)** the radian measure of 135°, and **(b)** the degree measure of $\frac{5}{2}\pi$ radians. Sketch each angle in a unit circle and find its sine, cosine, and tangent.

8.$^{\text{O}}$ **(a)** What is the length of the arc subtended on a circle of radius 14 meters by an angle of $\frac{7}{5}\pi$ radians with its vertex at the center of the circle? **(b)** What is the area of the sector of the circle inside the angle?

9.$^{\mathbf{O}}$ **(a)** Find the lengths, a and b, of the legs of the right triangle in Figure 48. Give exact and approximate decimal values. **(b)** Find the exact value of $\sqrt{a^2 + b^2}$.

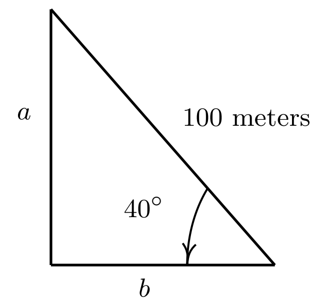


FIGURE 48

10.$^{\mathbf{O}}$ Use $\sqrt{2}$ and $\sqrt{3}$ as necessary to give exact values of **(a)** $\cos(\frac{3}{4}\pi)$, **(b)** $\sin(-\frac{1}{3}\pi)$, **(c)** $\tan(\frac{5}{4}\pi)$, and
(d) $\sec(\frac{1}{6}\pi)$.

11.$^{\mathbf{O}}$ What are the coordinates (x, y) of the point Q in Figure 49 if the points P and Q are 6 units apart?

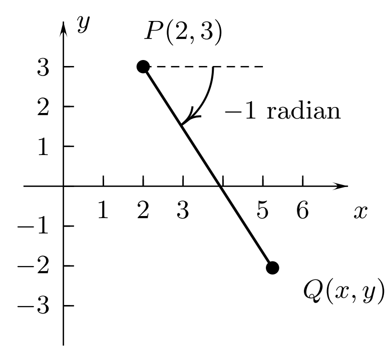


FIGURE 49

12.$^{\mathbf{O}}$ What is the angle γ in the right triangle of Figure 50?

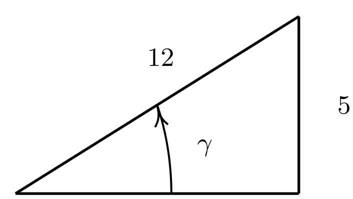


FIGURE 50

13.$^{\mathbf{O}}$ Use π as necessary to give exact values of **(a)** $\sin^{-1}(1/\sqrt{2})$, **(b)** $\tan^{-1}(-1)$, **(c)** $\sin^{-1}(-1)$, **(d)** $\tan^{-1}(-1/\sqrt{3})$, **(e)** $\sin^{-1}(\frac{1}{2}\sqrt{3})$, and **(f)** $\tan^{-1}(0)$.

14.[O] Give the exact and approximate decimal values of the angle α in Figure 51.

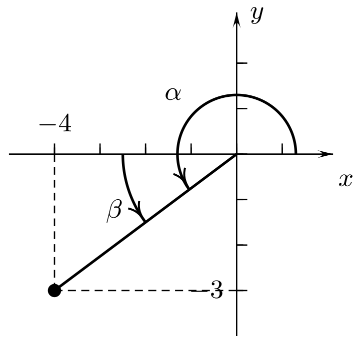


FIGURE 51

15.[O] The angle between the sides of lengths 8 and 10 feet in the triangle of Figure 52 is 0.9 radians. Find the length c of the other side.

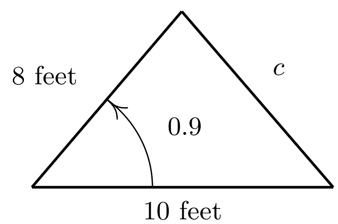


FIGURE 52

16.[O] What is the largest angle in a triangle with sides of lengths 3, 5, and 7?

17.[O] Two angles in a triangle are 0.5 and 1.8 radians. The side opposite the angle of 0.5 radians is 10 feet long. How long is the side opposite the angle of 1.8 radians?

18. Use $\sqrt{2}$ and $\sqrt{3}$ as needed to give the exact values of **(a**[A]**)** $\sin(\frac{7}{4}\pi)$, **(b**[A]**)** $\cos(\frac{2}{3}\pi)$, **(c**[A]**)** $\tan(-3\pi)$, **(d)** $\sec(\frac{4}{3}\pi)$, **(e**[A]**)** $\cot(\frac{1}{6}\pi)$, **(f)** $\tan(-\frac{5}{6}\pi)$, and **(g**[A]**)** $\sec(-\frac{3}{2}\pi)$.

19.[O] The largest ferris wheel of all time was designed by George Ferris and built for the 1893 World's Columbian Exposition in Chicago. It had a 125 foot radius and could carry 2160 passengers.[1] Suppose that the center of the wheel was 150 feet above the ground and that it turned counterclockwise as viewed in the schematic sketch with a uv-plane in Figure 53. The dot represents one of the cars on Ferris' wheel. Let $(u(\theta, v(\theta))$ be its coordinates when it is at the angle θ (radians) in Figure 52. **(a)** Which of Figures 54 and 55 shows the graph of $u = u(\theta)$ and which shows the graph of $v = v(\theta)$? **(b)** Give formulas for $u(\theta)$ and $v(\theta)$.

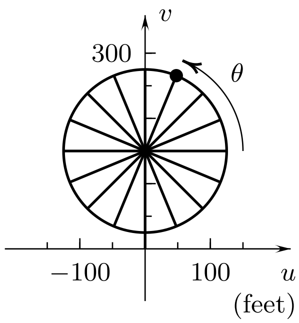


FIGURE 53

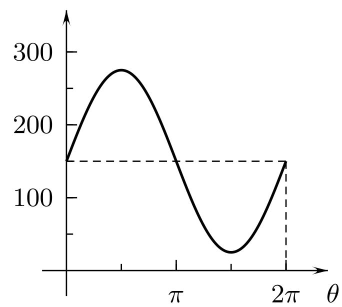


FIGURE 54

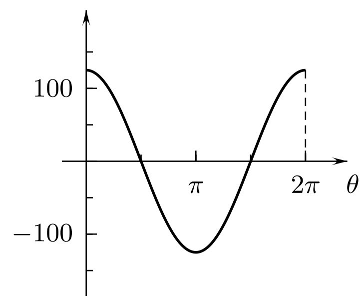


FIGURE 55

Give exact answers and approximate decimal values when appropriate in the following exercises.

20.[A] The Eiffel tower was designed by Alexandre Gustave Eiffel for the Paris exposition of 1889. Its base is a horizontal square 100 meters wide and a line from a corner of its base to its pointed top makes an angle of 1.34 radians. **(a)** How long are the diagonals of the base? **(b)** How high is the Eiffel tower?

21. While flying at an altitude of 5000 feet, Joan sees an airport ahead. Her line of sight makes an angle of 0.7 radians with the horizontal. How far is she from the airport? (Give the diagonal distance.)

21. Use π as needed to give exact values of **(a**[A]**)** $\sin^{-1}(-\frac{1}{2}\sqrt{2})$, **(b**[A]**)** $\sin^{-1}(0)$, **(c)** $\tan^{-1}(1)$, **(d)** $\sin^{-1}(\frac{1}{2})$, and **(e)** $\tan^{-1}(1/\sqrt{3})$.

22.[O] What is the smallest angle in the right triangle whose legs are 4000 and 1000 miles long?

23. A twelve-foot long ladder is leaning against a vertical wall. Its top is 8 feet above the level ground. What is the angle between the ladder and the ground?

24.[O] Ravenna, Italy is 73 kilometers north and 76 kilometers east of Florence (Figure 56). **(a)** How far is Ravenna from Florence? **(b)** What is the direction from Florence toward Ravenna?

FIGURE 56

[1] "The Ferris Wheel" by H. Petrovski, *American Scientist*, May-June, 1993, pp. 216–222.

25.[A] Panama City is 507 kilometers east and 102 kilometers south of San Jose, Costa Rica (Figure 57). **(a)** How far is San Jose from Panama City? **(b)** What is the direction from Panama City toward San Jose?

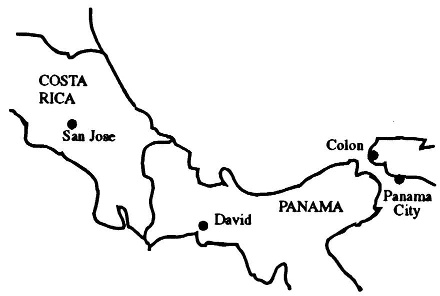


FIGURE 57

25A The city of David, Panama , is 210 miles from Panama City and the direction from David toward Panama City is 0.172 radians north of east (Figure 57). How far is Panama City north and east of David?

26. Beirut, Lebanon and Tel Aviv, Israel are 134 miles apart, and Tel Aviv is 42 miles west of Beirut (Figure 58). **(a)** How far is Haifa south of Beirut? **(b)** What is the direction from Beirut toward Tel Aviv?

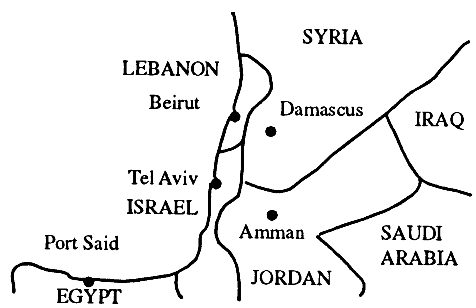


FIGURE 58

27. After an argument with Martha, George stomps out of the house and walks one-half mile east on one road and then one-quarter mile to the north on another road to a coffee shop. What are the angles in the triangle with vertices at his house, the coffee shop, and the intersection of the two roads?

EXPLORATION:

28.[O] Tony sails 3 miles directly south from his yacht club and then 8 miles toward the south-east. How far is he from the yacht club at that time?

29. Sketch the graphs of **(a**[O]**)** $y = -\cos(2x + \frac{1}{4}\pi)$, **(b**[O]**)** $y = 2\sin(x + \frac{1}{6}\pi)$, **(c)** $y = 10\cos(\frac{1}{3}x)$, and **(d)** $y = \sin(\frac{1}{4}(x - \pi))$.

30.[A] The earth's orbit about the sun is approximately a circle of radius 93 million miles. How far does the earth move in its orbit in a day?

31. An angle of 4 radians with its vertex at the center of a circle subtends an arc 10 feet long on the circle. What is the radius of the circle?

32.[A] When the Great Pyramid of Egypt was built in the fourth dynasty (c. 2600–2500 BC), it was a right pyramid 481 feet high with a square base 776 wide. What angle did its sides make with the ground? (Calculate the angle at the middle of one side of the base.)

33.[O] Two sides of a triangle are 5 and 7 feet long and the angle between them is $\frac{1}{3}\pi$. How long is the other side?

34.[A] The town of Jalapeño is 20 kilometers north-northeast of the town of Ancho, and the town of Habanero in 30 kilometers southwest of Ancho. How far is it from Jalapeño to Habanero?

35. The angle γ in the isosceles triangle of Figure 59 is $\frac{3}{4}\pi$. **(a)** What are the other angles? **(b)** How long is the unequal side?

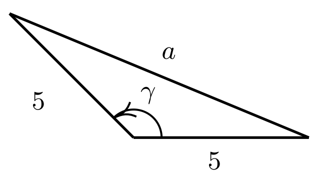


FIGURE 59

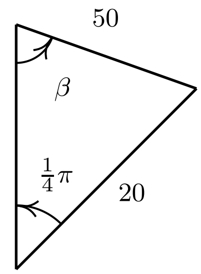


FIGURE 60

36.[O] What is the angle β in Figure 60?

37.[O] What is the obtuse angle in the triangle with vertices $(1,0),(0,0)$, and $(-1,1)$ in an xy-plane with equal scales on the axes?

38.[A] The Bermuda Triangle is an area of the Atlantic Ocean where many ships and airplanes have disappeared. The vertices of the triangle are in Bermuda, in Puerto Rico, and at Melbourne, Florida.(2) Melbourne is 1175 miles from Bermuda and 1000 miles from Puerto Rico, and it is 975 miles from Bermuda to Puerto Rico. What is the smallest angle in the Bermuda Triangle?

39. Julie walks 400 meters in one direction, turns and walks 300 meters in another direction, turns and walks 200 meters in a third direction, and ends up where she started. **(a)** How far does she walk? **(b)** What is the largest angle in the triangle formed by her path?

40.[O] Surveyers who want to measure the distance from point A, were they are standing to a point B across a river, pick a point C 100 meters away on their same side of the river and measure the angles at A and C in the triangle ABC. They find that the angle at A is 1.9 radians and the angle at C is 0.8 radians. How far is it from A to B?

41.[A] The moon's diameter is approximately 2,000 miles and its distance from the earth varies between 226,000 and 252,000 miles. **(a)** What are the approximate maximum and minimum angles of vision subtended by the moon when viewed from the earth? **(b)** As viewed from the earth the moon moves relative to the stars because its phases have a period of 29.53 days. By approximately what multiple of its diameter does the moon move every hour relative to the stars?

C **42.** Let s denote the length of the side opposite the angle θ in a triangle whose other sides are of length 3 and 5. Give a formula for s as a function of θ and generate its graph for $0 \le \theta \le \pi, 0 \le s \le 10$. Copy it on your paper and explain its shape.

(2) *The World Book Encyclopedia*, Volume 2, Chicago: World Book-Childcraft International, Inc., 1978, p. 208.

Find all solutions x of the equations in Exercises 43 through 56. Express the answers without using any inverse trigonometric functions.

43.[O] $\sin(5x) = -\frac{1}{2}\sqrt{2}$

44.[O] $\cos(\frac{1}{7}x) = -1$

45.[O] $\sin^2 x = \frac{1}{2}$

46.[A] $\sin(x^2) = \frac{1}{2}$

47. $\cos(3x + 1) = 0$

48. $\cos^3(2x) = -\frac{1}{8}$

49. $\sin^2 x - \sin x = 0$

50.[O] $\sec(3x) = \sqrt{2}$

51. $\csc(-x) = 2$

52.[O] $\tan(x/5) = 1/\sqrt{3}$

53.[A] $\tan^3 x = 3\tan x$

54. $\cot x = \tan x$

55. $\sec^2 x + \tan^2 x = 3$

56. $\tan x + \sec x = 0$

(End of Section 0.5)