Section 0.6

(3/31/07)

Linear combinations, products, quotients, and compositions

OVERVIEW: *Many funcions that are used in calculus are constructed from constant, power, exponential, logarithmic, trigonometric, and inverse trigonometric functions by taking linear combinations, products, quotients, and compositions of these basic functions. In this section we discuss these procedures and look at instances where the graphs of such functions can be readily explained by their formulas.*

Topics:

- ***Linear combinations***
- ***Products and quotients***
- ***Polynomials and rational functions***
- ***Composite functions***

Linear combinations

A LINEAR COMBINATION of functions is formed by multiplying each of the functions by a constant and adding the results.

Example 1 The functions **(a)** $A(x) = \frac{1}{8}x^2 + 3\cos x$, **(b)** $B(x) = \frac{1}{8}x^2 - 3\cos x$, and **(c)** $C(x) = -\frac{1}{8}x^2 + 3\cos x$ are linear combinations of $y = x^2$ and $y = \cos x$. Match them with their graphs in Figures 1 through 3.

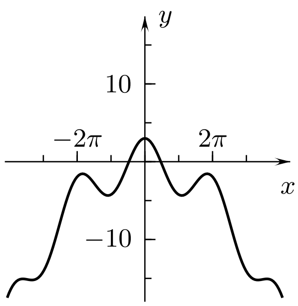


FIGURE 1

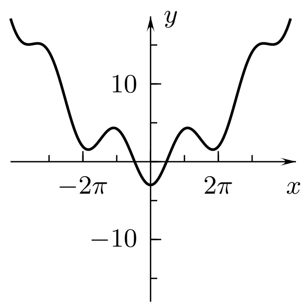


FIGURE 2

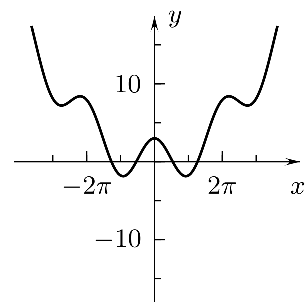


FIGURE 3

SOLUTION **(a)** The curves $y = \frac{1}{8}x^2$ and $y = 3\cos x$ are shown in Figure 4. The first is $y = x^2$ contracted vertically by a factor of 8, and the second is $y = \cos x$ magnified vertically by a factor of 3. Because $y = 3\cos x$ oscillates between 3 and -3, the graph of $A(x) = \frac{1}{8}x^2 + 3\cos x$ oscillates from 3 units above $y = \frac{1}{8}x^2$ to 3 units below it, and since $A(0) = \frac{1}{8}(0)^2 + 3\cos(0) = 3$, the graph is in Figure 3.

(b) The graph of $B(x) = \frac{1}{8}x^2 - 3\cos x$ oscillates from 3 units above to 3 units below $y = \frac{1}{8}x^2$ and $B(0) = \frac{1}{8}(0)^2 - 3\cos(0) = -3$. The graph is in Figure 2.

(c) Figure 5 shows the curves $y = -\frac{1}{8}x^2$ and $y = 3\cos x$. The graph of $C(x) = -\frac{1}{8}x^2 + 3\cos x$ oscillates from 3 units above to 3 units below $y = -\frac{1}{8}x^2$. It is in Figure 1. □

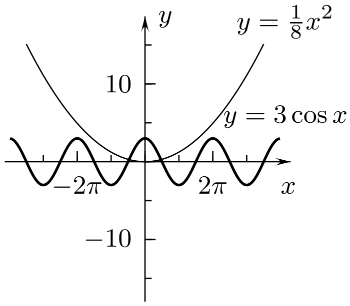


FIGURE 4

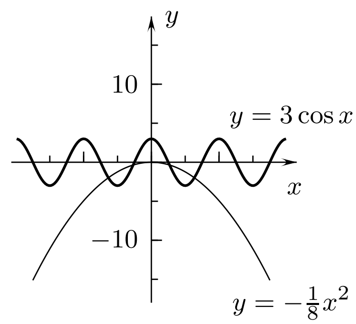


FIGURE 5

C ***Question 1*** Generate $y = \frac{1}{8}x^2 + 10\cos x$ in the window $-4\pi \le x \le 4\pi, -15 \le y \le 20$ with x-scale $= \pi$ and y-scale $= 5$ and explain how this curve differs from $y = \frac{1}{8}x^2 + 3\cos x$ in Figure 3.

The domain of a linear combination of functions consists of those values of the variable x where all of the functions involved are defined.

Example 2 What is the domain of $g(x) = 2 + 3x^{1/2} - 5^x$?

SOLUTION Because the constant function $y = 2$ and the exponential function $y = 5^x$ are defined for all x, the domain of $g(x) = 2 + 3x^{1/2} - 5^x$ is the interval $[0, \infty)$ where the square root function $y = x^{1/2}$ is defined. □

Products and quotients

The PRODUCT of two functions f and g is the function fg whose value at x is the product $f(x)g(x)$ of the values of the two functions. Its domain consists of all values of x where $f(x)$ and $g(x)$ are defined.

The QUOTIENT f/g has the value $f(x)/g(x)$ at x. Its domain consists of all x such that $f(x)$ and $g(x)$ are defined and $g(x)$ is not zero.

Example 3 What is the domain of $y = \dfrac{x}{\ln x}$?

SOLUTION Because $y = x$ is defined for all x, while $y = \ln x$ is defined for $x > 0$ and is 0 at $x = 1$, the domain of $y = \dfrac{x}{\ln x}$ consists of all positive $x \ne 1$. The domain is the union $(0,1) \cup (1,\infty)$ of two open intervals. This is illustrated by the graph of the function in Figure 6, which also shows the domain on the x-axis. □

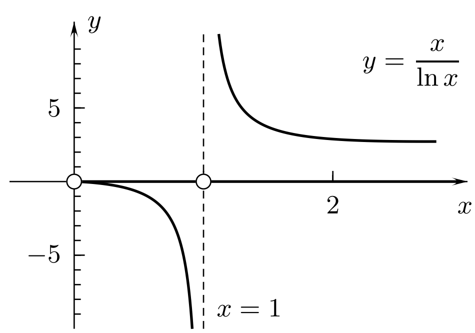


FIGURE 6

Question 2 How does the domain of $y = x\ln x$ differ from the domain of $y = \dfrac{x}{\ln x}$?

Polynomials and rational functions

POLYNOMIALS are linear combinations of constants and power functions $y = x^n$ with positive integer exponents. The function $F(x) = 3 - 5x^3 + x^6$, for example, is a polynomial. RATIONAL FUNCTIONS are linear combinations of polynomials and quotients of polynomials.

Example 4 The functions **(a)** $y = \frac{1}{2}x^3 + 4$, **(b)** $y = \dfrac{1}{x^2+1}$, and **(c)** $y = \dfrac{x}{(x-1)^2}$ are rational functions. Match them to their graphs in Figures 7 through 9. Give reasons for your choices.

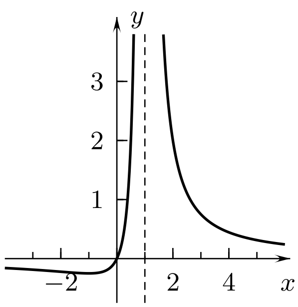


FIGURE 7

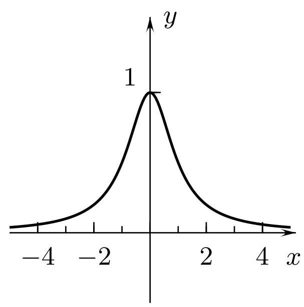


FIGURE 8

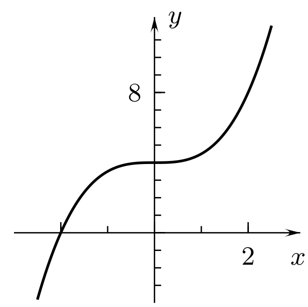


FIGURE 9

SOLUTION **(a)** Because $y = \frac{1}{2}x^3 + 4$ is obtained from $y = x^3$ by contracting it vertically by a factor of 2 and then raising it 4 units, this curve is in Figure 9.

(b) The function $y = \dfrac{1}{x^2+1}$ is defined and positive for all x. Its value is 1 at $x = 0$, is between 0 and 1 for $x \neq 0$, and is small for large positive and large negative x. Therefore, its graph is in Figure 8.

(c) Because of the term $x - 1$ in the denominator, $y = \dfrac{x}{(x-1)^2}$ is not defined and its graph has a vertical asymptote at $x = 1$. It is the curve in Figure 7. □.

Composite functions

If we apply the function g to the variable x to obtain $g(x)$, and then apply the function f, we get the number $f(g(x))$:

$$x \xrightarrow{g} g(x) \xrightarrow{f} f(g(x)).$$

The function that combines these actions is called the COMPOSITION of f and g. It is denoted $f \circ g$ (read "f composed with g" or "f circle g"):

$$x \xrightarrow{f \circ g} f(g(x)).$$

Here is a general definition:

Definition 1 *The value of the* COMPOSITE FUNCTION $f \circ g$ *at* x *is the value of* f *at* $g(x)$*:*

$$(f \circ g)(x) = f(g(x)). \tag{1}$$

The domain of $f \circ g$ consists of all numbers x in the domain of g such that $g(x)$ is in the domain of f.

Example 5 What are the values of $y = (F \circ G)(x)$ and $y = (G \circ F)(x)$ at $x = 2$ if $F(x) = x^2$ and the values of $y = G(x)$ are given in the table below?

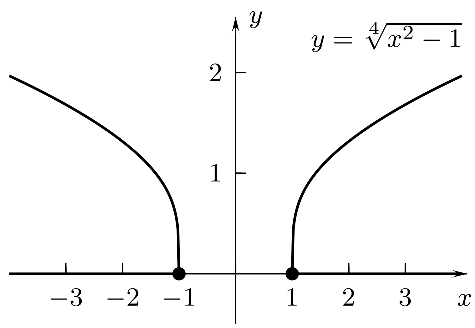
x	0	1	2	3	4
$G(x)$	15	10	5	0	-5

SOLUTION Because $F(x) = x^2$, the value of $y = (F \circ G)(x)$ at $x = 2$ is $F(G(2)) = [G(2)]^2$. The table gives $G(2) = 5$, so $F(G(2)) = 5^2 = 25$. Similarly, the value of $y = (G \circ F)(x)$ at $x = 2$ is $G(F(2)) = G(2^2) = G(4) = -5$. □

Example 6 **(a)** What is $(f \circ g)(x)$ for $f(x) = \sqrt[4]{x}$ and $g(x) = x^2 - 1$? **(b)** What is the domain of $f \circ g$ in this case?

SOLUTION **(a)** There are two ways to obtain the formula. We can write $(f \circ g)(x) = f(g(x)) = \sqrt[4]{g(x)} = \sqrt[4]{x^2 - 1}$, or we can write $(f \circ g)(x) = f(x^2 - 1) = \sqrt[4]{x^2 - 1}$.

(b) Because $g(x) = x^2 - 1$ is defined for all x, the domain of $f \circ g$ consists of all x such that $g(x) = x^2 - 1$ is in the domain of $f(x) = \sqrt[4]{x}$. Then, since $f(x)$ is defined for $x \geq 0$, the domain of $f \circ g$ is the set of all numbers x such that $x^2 - 1 \geq 0$. Adding 1 to both sides gives the equivalent inequality $x^2 \geq 1$, and then taking square roots of both sides gives $|x| \geq 1$. The domain is the union $(-\infty, -1] \cup [1, \infty)$ of two infinite closed intervals. This is illustrated by the graph of $f \circ g$ in Figure 10. □



$(f \circ g)(x) = \sqrt[4]{x^2 - 1}$

FIGURE 10

C ***Question 3*** Give a formula for $g \circ f$ with $f(x) = \sqrt[4]{x}$ and $g(x) = x^2 - 1$ from Example 6 and generate its graph in the window $-0.5 \leq x \leq 5, -1.75 \leq y \leq 1.75$. What is the domain of this function?

Example 7 **(a)** What are the domains of $p \circ q$ and $q \circ p$ for $p(x) = e^x$ and $q(x) = \sin x$?
(b) Match the functions $p \circ q$ and $q \circ p$ to their graphs in Figures 11 and 12. Explain how you make your choices.

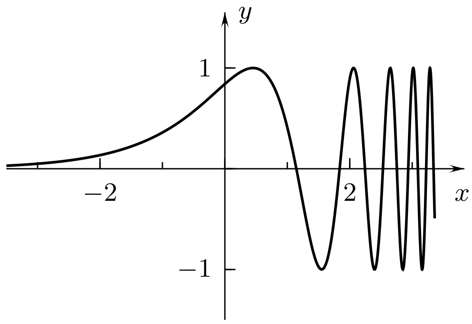


FIGURE 11

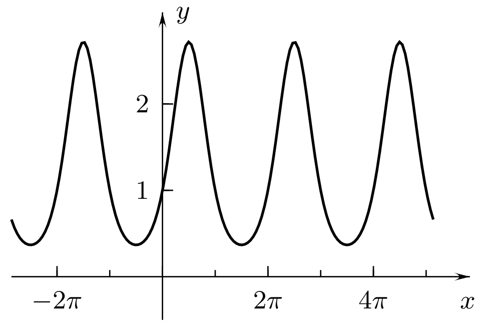


FIGURE 12

SOLUTION **(a)** The composite functions $y = (p \circ q)(x)$ and $y = (q \circ p)(x)$ are defined for all x because $p(x) = e^x$ and $q(x) = \sin x$ are defined for all x.
(b) In this case we can distinguish between the two composite functions by finding their ranges. The values of $(p \circ q)(x) = e^{\sin x}$ oscillate between $e^1 = e \doteq 2.718282$ and $e^{-1} = 1/e \doteq 0.367879$ because $\sin x$ oscillates between 1 and -1. Its range is $[e^{-1}, e]$ and its graph is in Figure 12. The values of $(q \circ p)(x) = \sin(e^x)$ oscillate between 1 and -1. Its range is $[-1, 1]$ and its graph is in Figure 11. (We could also recognize that the graph of $y = e^{\sin x}$ is in Figure 12 because it has period 2π and the function in Figure 11 is not periodic.) □

Example 8 Define functions f, g, and h such that $(f \circ g \circ h)(x) = \sqrt{\ln(x^4+1)}$.

SOLUTION Notice that values of $\sqrt{\ln(x^4+1)}$ are calculated from the inside out. First $x^4 + 1$ is calculated, then this number is used to find $\ln(x^4+1)$, and finally the last number is used to calculate $\sqrt{\ln(x^4+1)}$. Similarly, values of $(f \circ g \circ h)(x) = f(g(h(x)))$ are calculated from the inside out. Consequently, we let h be the innermost function by setting $h(x) = x^4 + 1$, let g be the intermediate function by defining $g(x) = \ln x$, and have f be the outermost function by writing $f(x) = \sqrt{x}$. Then

$$\begin{aligned}(f \circ g \circ h)(x) &= f(g(h(x)) = f(g(x^2+1)) \\ &= f(\ln(x^4+1)) = \sqrt{\ln(x^4+1)}. \ \square\end{aligned}$$

For the composite functions discussed above, all functions involved had the same variable x. In many applications different letters are used for the variables of different functions, as is illustrated in the next example.

Example 9 Suppose that a balloon is $h = h(t)$ feet above the ground at time t minutes for $0 \le t \le 15$ and that the volume of the balloon is $V = V(h)$ cubic feet when it is h feet above the ground. **(a)** What is the significance of the composite function $V \circ h$ and what is its domain? **(b)** What is $(V \circ h)(10)$ if $h(10) = 500$ feet and $V(500) = 35$ cubic feet?

SOLUTION **(a)** $(V \circ h)(t) = V(h(t))$ is the volume of the balloon at time t and is defined for $0 \le t \le 15$.
(b) $(V \circ h)(10) = V(h(10)) = V(500) = 35$ cubic feet. □

Question 4 A company receives $R = R(x)$ dollars of revenue when it sells x units of an item. It makes $P = P(R)$ dollars profit on R dollars revenue. It pays $T = T(P)$ dollars tax on a profit of P dollars. What is the significance of the composite function $T \circ P \circ R$?

Responses 0.6

Response 1 Figure R1 • $y = \frac{1}{8}x^2 + 10\cos x$ oscillates from 10 units above to 10 units below $y = \frac{1}{8}x^2$ instead of from 3 units above to 3 units below $y = \frac{1}{8}x^2$ like $y = \frac{1}{8}x^2 + 3\cos x$.

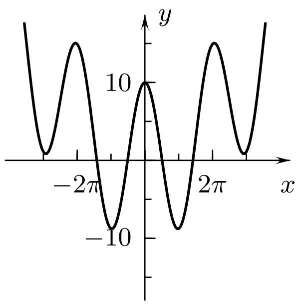


Figure R1

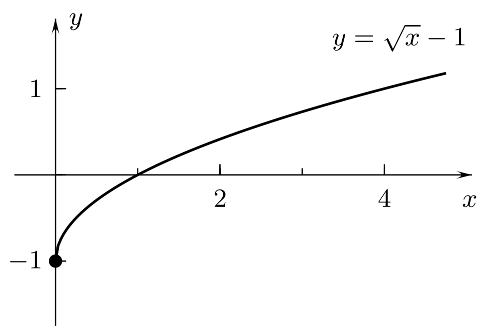


Figure R3

Response 2 The domain of $y = x\ln x$ is the entire domain $(0, \infty)$ of $y = \ln x$ because $y = x$ is defined for all x and $\ln x$ is not in the denominator, as it is for $y = x/\ln x$.

Response 3 $(g \circ f)(x) = (\sqrt[4]{x})^2 - 1 = \sqrt{x} - 1$ • Figure R3 • This function is defined for $x \geq 0$, where $f(x) = \sqrt[4]{x}$ is defined since $g(x) = x^2 - 1$ is defined for all x.

Response 4 $(T \circ P \circ R)(x) = T(P(R(x)))$ is the tax on x units of the item that are sold.

Interactive Examples 0.6

Interactive solutions are on the web page http//www.math.ucsd.edu/˜ashenk/.[†]

1. **(a)** Give a formula for $y = (B \circ z)(x)$ where $B(x) = x/(x-2)$ and $z(x) = \sqrt{x}$. **(b)** What is the domain of $B \circ z$?

2. What is $Z(4)$ if $Z(x) = [(W \circ Y)(x)]^3, Y(4) = 2$, and $W(2) = 10$?

3. Express $y = 3\cos^4 x - 2\cos^2 x$ as a composition $(f \circ g)(x) = f(g(x))(x)$ of functions $y = f(x)$ and $y = g(x)$ with simpler formulas.

Exercises 0.6

[A]Answer provided. [O]Outline of solution provided. [C]Graphing calculator or computer required.

CONCEPTS:

1. How are the functions $f + g$ and $g + f$ related?

2. How are the functions fg and gf related?

3. How are the functions f/g and g/f related if $f(x) = e^x$ and $g(x) = x^2 + 1$?

4. How are the functions $A \circ B$ and $B \circ A$ related if $A(x) = x^3$ and $B(x) = x^5$?

5. Show that for $P(x) = e^x$ and $Q(x) = x^2$, $(P \circ Q)(x)$ and $(Q \circ P)(x)$ are equal only at $x = 0$ and $x = 2$.

[†]In the published text the interactive solutions of these examples will be on an accompanying CD disk which can be run by any computer browser without using an internet connection.

BASICS:

6. The lower curve in Figure 13 is the graph of $y = R(x)$ and the upper curve is the graph of $y = S(x)$. Sketch the graph of $y = R(x) + S(x)$.

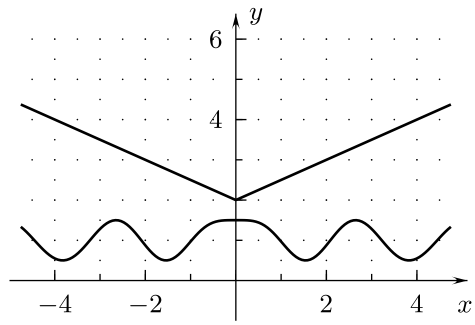


FIGURE 13

7.[O] Use the formula for the function $y = x\sin(\pi x)$ to explain the shape of its graph in Figure 14.

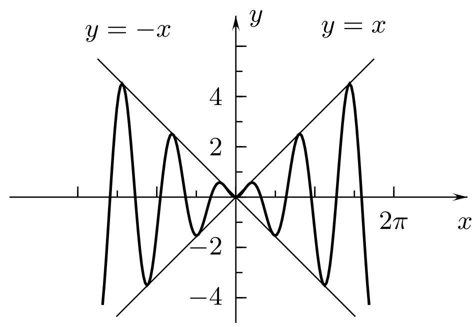


FIGURE 14

8.[O] What are the domains of **(a)** $y = 2x - \sqrt[3]{x} + 3x^{-1}$, **(b)** $y = 5\log_{10} x + 10\ln x$, and **(c)** $y = \sin x + \sin^{-1} x$?

9.[O] **(a)** For what values of x is $y = 2\tan x + 3\sec x$ defined? **(b)** For what values of x is $y = \cot x - \csc x$ defined?

10.[O] What is the domain of $y = \dfrac{\sin^{-1} x}{x^2}$?

11.[O] What are the values at $x = 9$ of **(a)** $y = Q(\sqrt{x})$, **(b)** $y = \sqrt{Q(x)}$, and **(c)** $y = Q(x) + \sqrt{x}$ if $Q(3) = 10$ and $Q(9) = 36$?

12.[O] What are **(a)** $(M \circ N)(5)$, **(b)** $(N \circ M)(5)$, and **(c)** $(M \circ M)(5)$ if $M(5) = 10, N(5) = 20, M(10) = 30, N(10) = 40, M(20) = 50$, and $N(20) = 60$?

13.[O] What are the domains of **(a)** $y = \sin(\ln x)$ and **(b)** $y = \ln(\sqrt{x})$?

14.[A] What are the values of **(a)** $5f + 3g$ **(b)** fg, and **(c)** f/g at $x = 10$ if $f(10) = -20$ and $g(10) = 5$?

15. What is the value of $y = p(x+1)p(x-1)$ at $x = 2$ if $p(1) = 3$ and $p(3) = 10$?

16.[O] What is $H(4)$ if $H(x) = J(r(x)), r(4) = -1$, and $J(-1) = 10$?

17. **(a)** Give a formula for $y = U(V(x))$ where $U(V) = V^2/(V+1)$ and $V(x) = x^3$. **(b)** What is the domain of $y = U(V(x))$?

18. **(a)** Give a formula for $y = (R \circ x)(t)$ where $R(x) = x\sqrt{30 - 5x}$ and $x(t) = t + 5$. **(b)** What is the domain of $R \circ x$?

19.[A] **(a)** Give a formula for $y = (T \circ K)(x)$ where $T(x) = x + x^{-1/2}$ and $K(x) = x^2 - 4$. **(b)** What is the domain of $T \circ K$?

20. What is $A(B(100))$ if $B(100) = 1000$ and $A(1000) = 14$?

21. What is the value of $y = [F(x^3)]^2$ at $x = -1$ if $F(-1) = 7$?

22. The graphs of $y = L(x)$ and $y = M(x)$ are in Figures 15 and 16. Does Figure 17 show the graph of $y = L(M(x))$ or of $y = M(L(x))$?

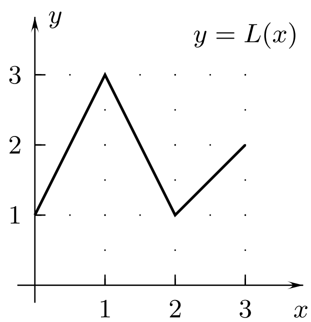


FIGURE 15

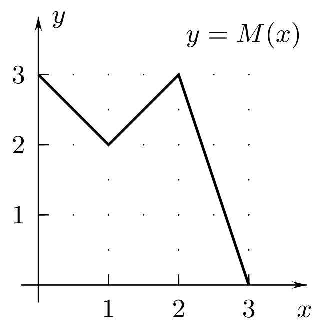


FIGURE 16

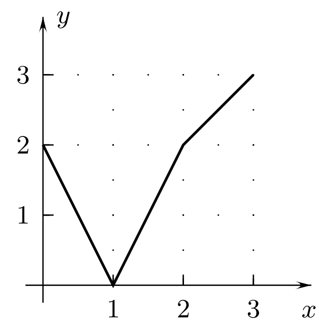


FIGURE 17

Express the functions in Exercises 23 through 28 as compositions $(f \circ g)(x) = (f\,(g(x))\,(x)$ of functions $y = f(x)$ and $y = g(x)$ with simpler formulas.

23.[O] $y = \sin(x^2)$

24.[A] $y = \sin^2 x$

25. $y = \ln(e^x + 7)$

25.[O] $y = e^{\sin x} + 1$

26. $y = \cos(x^5) + \sin(x^5)$

EXPLORATION:

27. Figures 18 and 19 show the graphs of two functions $y = F(x)$ and $y = G(x)$. Match the functions **(a**[O]**)** $y = F(x) + G(x)$, **(b**[A]**)** $y = F(x) - G(x)$, **(c)** $y = F(x)G(x)$, and **(d)** $y = F(x)/G(x)$ to their graphs in Figures 20 through 23. Justify your choices.

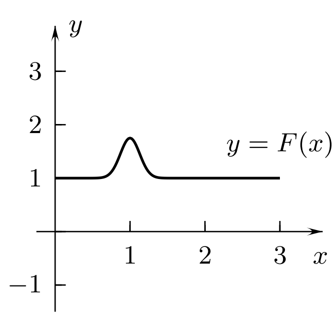


FIGURE 18

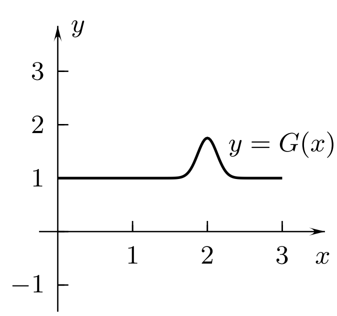


FIGURE 19

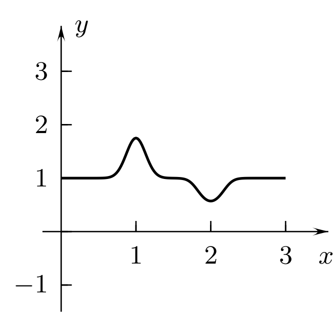


FIGURE 20

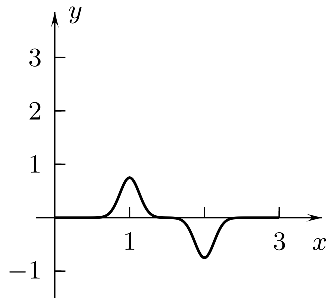


FIGURE 21

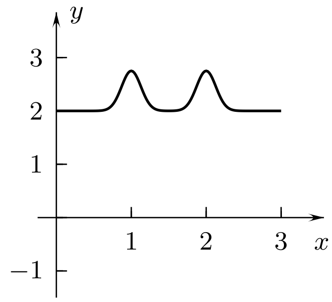


FIGURE 22

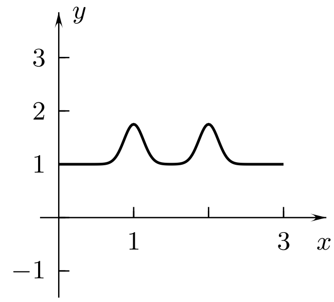


FIGURE 23

28. Figures 24 through 26 show the graphs of **(a[O])** $y = x\cos(5x)$, **(b)** $y = x + \cos(5x)$, and **(c)** $y = \dfrac{\cos(5x)}{x}$. Match the functions to the graphs. Justify your answers.

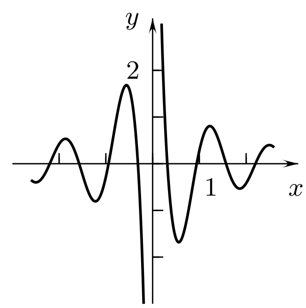


FIGURE 24

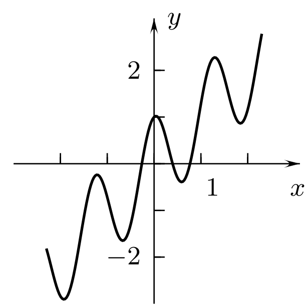


FIGURE 25

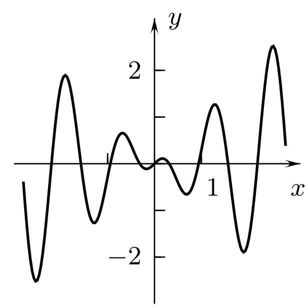


FIGURE 26

29. Match the functions **(a[O])** $y = x^3 - \dfrac{1}{x}$, **(b)** $y = x^2 - \dfrac{1}{x^2}$, and **(c)** $y = 2x + \dfrac{1}{x}$ to their graphs in Figures 27 through 29. Justify your answers.

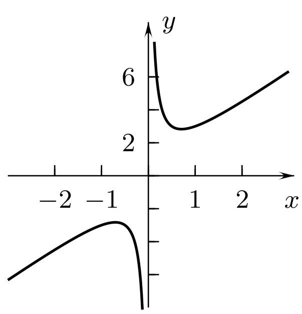


FIGURE 27

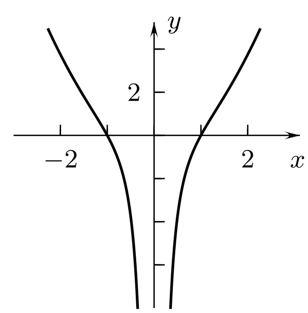


FIGURE 28

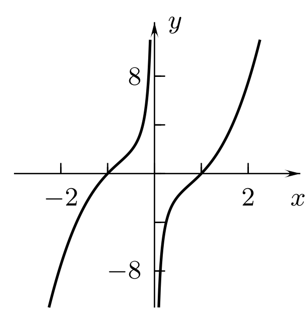


FIGURE 29

30.[A] What are the domains **(a)** of $y = f(x^3)$ and **(b)** of $y = [f(x)]^3$ if the domain of f is $[0, 8]$?

31.[A] Match the functions **(a)** $y = \ln(x^2 - 1)$, **(b)** $y = \ln(x^2 + 1)$, and **(c)** $y = \ln(1/x)$ to their graphs in Figures 30 through 32. Justify your conclusions.

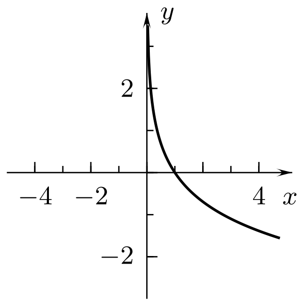


FIGURE 30

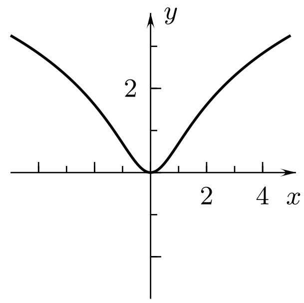


FIGURE 31

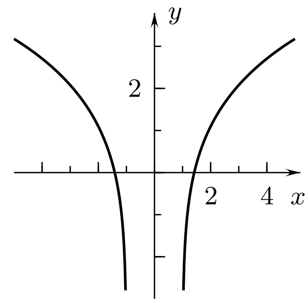


FIGURE 32

32. Match the functions **(a**[O]**)** $y = \ln(\cos x)$, **(b)** $y = \ln(\sec x)$, and **(c)** $y = \ln(\cos x + 4)$ to their graphs in Figures 33 through 35. Justify your conclusions.

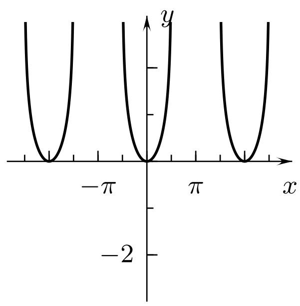


FIGURE 33

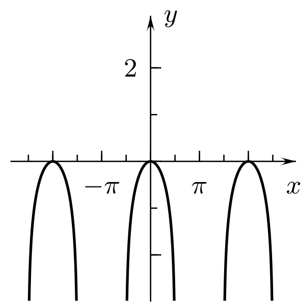


FIGURE 34

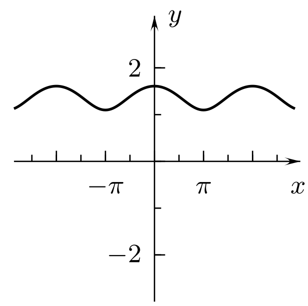


FIGURE 35

33. Figures 36 through 38 show the graphs of **(a**[O]**)** $y = \tan^2 x$, **(b)** $y = \tan(x^2)$, and **(c)** $y = x\tan x$. Match the functions to the graphs and explain how the formulas determine the shapes of the curves.

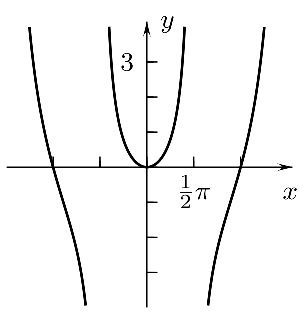


FIGURE 36

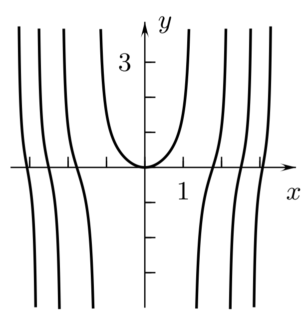


FIGURE 37

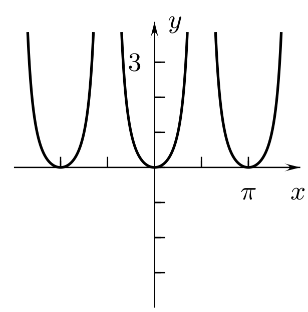


FIGURE 38

34. Figures 39 through 42 show the graphs of **(a$^{\text{O}}$)** $y = \sin^2 x$, **(b)** $y = \sqrt{\sin x}$, **(c)** $y = \sin^3 x$, and **(d)** $y = \sqrt[3]{\sin x}$. Match the functions to the graphs and explain how the formulas determine their shapes.

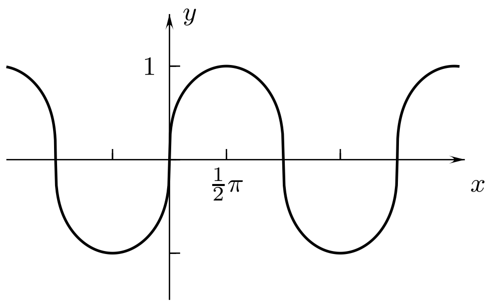


FIGURE 39

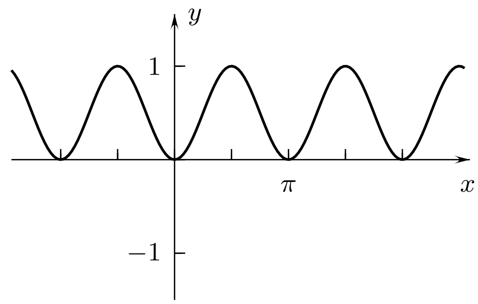


FIGURE 40

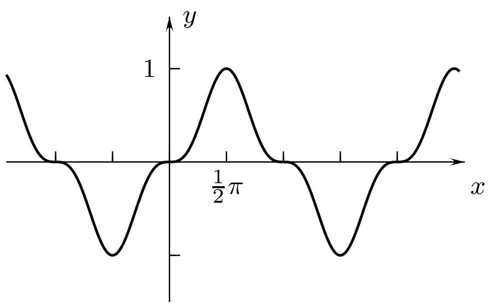


FIGURE 41

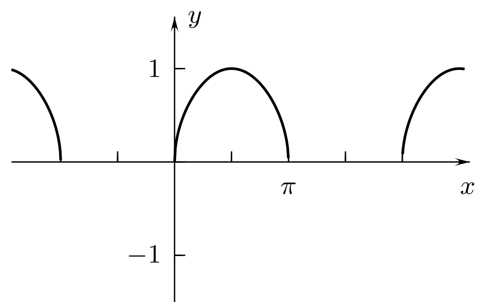


FIGURE 42

C **35.** Generate **(a)** $y = \sin(\sin^{-1} x)$, **(b)** $y = \cos(\cos^{-1} x)$, and **(c)** $y = \tan(\tan^{-1} x)$ separately in a window with equal scales on the axes and that contains the square $-2 \le x \le 2, -2 \le y \le 2$. Copy the curves on your paper and explain the results.

C **36.** Generate $y = \sin^{-1} x + \cos^{-1} x$ in the window $-2 \le x \le 2, -0.5 \le y \le 2.5$. Copy it on your paper and explain its shape.

37. Figures 43 and 44 show the graphs of **(a)** $y = \sin^{-1}(\sin x)$ and **(b)** $y = \cos^{-1}(\cos x)$. Match the graphs to their equations and explain their shapes.

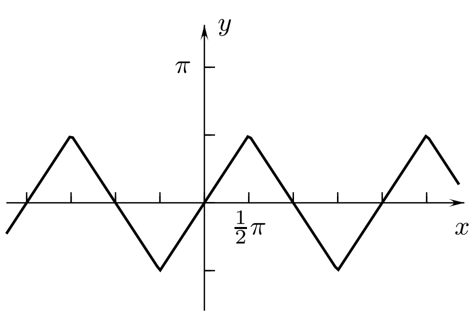


FIGURE 43

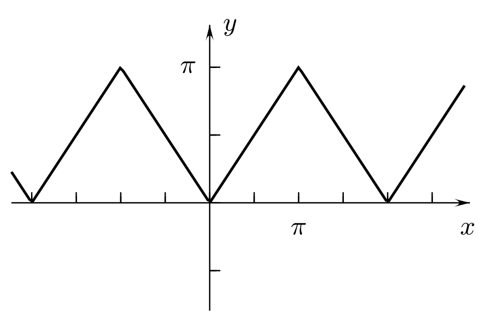


FIGURE 44

38. Is the curve in Figure 45 the graph of $y = \sin(\cos^{-1} x)$ or of $y = \cos(\sin^{-1} x)$? Explain.

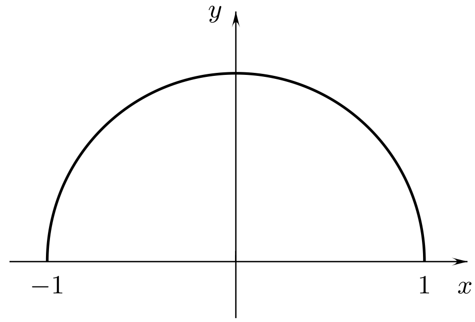


FIGURE 45

C **39.** Figure 46 shows the graphs of four functions given by $y = \tan^{-1}(Ax + B)$ with integer values of A and B. What are the four functions? Check your answer by generating the four graphs in the window $-6 \le x \le 6, -1.75 \le y \le 1.75$.

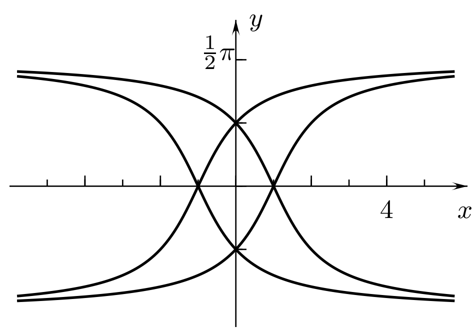


FIGURE 46

(End of Section 0.6)