Section 14.3

(3/23/08)

Partial derivatives with two variables

OVERVIEW: *In this section we begin our study of the calculus of functions with two variables. Their derivatives are called partial derivatives and are obtained by differentiating with respect to one variable while holding the other variable constant. We describe the geometric interpretations of partial derivatives, show how formulas for them can be found with differentiation formulas with one variable, and demonstrate how they can be estimated from tables and level curves.*

Topics:

- ***Limits of functions with two variables***
- ***Continuity of functions with two variables***
- ***Partial derivatives***
- ***A geometric interpretation of partial derivatives***
- ***Estimating partial derivatives from tables***
- ***Estimating partial derivatives from level curves***

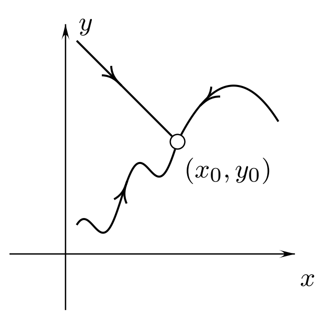
Limits of functions with two variables

In studying functions of one variable we used one- and two-sided limits. We cannot talk of two-sided or one-sided limits of functions of two variables. Instead we find limits by studying the values of functions along paths, as in the next definition.†

Definition 1 *Suppose that the function $z = f(x,y)$ is defined in a circle with its center at the point (x_0, y_0), except possibly at the point (x_0, y_0) itself. Then the limit of $f(x,y)$ as (x,y) approaches (x_0, y_0) is L and we write*

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \tag{1}$$

if the number $f(x,y)$ approaches L as (x,y) approaches (x_0,y_0) along all paths that lie in the circle and do not contain the point (x_0,y_0) (Figure 1). Here L can be a number or $\pm\infty$.



Three paths to (x_0, y_0)

FIGURE 1

†The formal definition of this limit for numbers L reads as follows: The limit of $f(x,y)$ is L as $(x,y) \to (x_0,y_0)$ if for each $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x,y) - L| < \epsilon$ for all points $(x,y) \neq (x_0,y_0)$ within a distance δ of (x_0,y_0). The definitions for $L = \pm\infty$ are similar.

Example 1 What is $\lim_{(x,y)\to(3,2)} (x^2 + y^2)$?

SOLUTION As $(x, y) \to (3, 2)$, the number x tends to 3 and the number y tends to 2. Then, because $A(x) = x^2$ is continuous for all x and $B(y) = y^2$ is continuous for all y, $x^2 \to 3^2$ and $y^2 \to 2^2$, so that

$$\lim_{(x,y)\to(3,2)} (x^2 + y^2) = 3^2 + 2^2 = 9 + 4 = 13. \square$$

Example 2 What is the limit of $z = \dfrac{1}{\sqrt{x^2+y^2}}$ as $(x, y) \to (0, 0)$?

SOLUTION Because $\sqrt{x^2+y^2}$ is positive for $(x, y) \neq (0, 0)$ and tends to 0 as $(x, y) \to (0, 0)$,

$$\lim_{(x,y)\to(0,0)} \frac{1}{\sqrt{x^2+y^2}} = \infty. \square$$

The result of Example 2 is illustrated in Figure 2, which shows the graph of $z = \dfrac{1}{\sqrt{x^2+y^2}}$. The z-coordinates of points on the surface tend to ∞ as their x- and y-coordinates tend to zero.

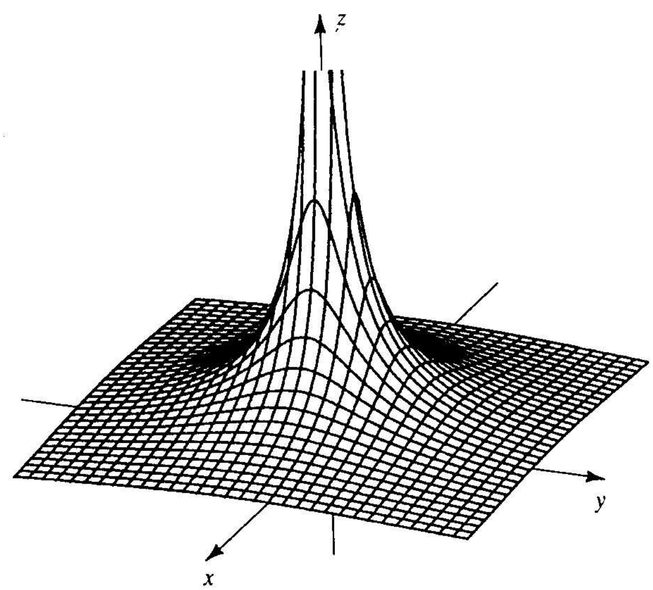


FIGURE 2

Continuity of functions with two variables

The definition of continuity for functions of two variables is similar to the definition for functions of one variable.

Definition 2 (Continuity) *A function $z = f(x, y)$ is continuous at a point (x_0, y_0) if it is defined in a circle centered at (x_0, y_0) and* $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$.

Any function $z = f(x, y)$ given by one formula that is constructed from the basic functions of one variable by adding, multiplying, dividing, and composition is continuous at any point such that it is defined in a circle centered at the point.

Partial derivatives

The PARTIAL DERIVATIVES of a function $z = f(x, y)$ of two variables are defined as follows.

Definition 3 (Partial derivatives) *The x-PARTIAL DERIVATIVE (or x-DERIVATIVE) and y-PARTIAL DERIVATIVE (or y-DERIVATIVE) of $z = f(x, y)$ at (x, y) are the limits,*

$$\left.\frac{\partial f}{\partial x}\right|_{(x,y)} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} \tag{2}$$

$$\left.\frac{\partial f}{\partial y}\right|_{(x,y)} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} \tag{3}$$

provided these limits exist and are finite.

The derivatives in this definition are also denoted $f_x(x, y)$ and $f_y(x, y)$ and are referred to as the FIRST DERIVATIVES or FIRST-ORDER DERIVATIVES of f.

Definition **(2)** is the same as the definition from Chapter 2 of the x-derivative of $f(x, y)$ viewed as a function of x. Similarly definition **(3)** is the same as the definition of the y-derivative of $f(x, y)$ viewed as a function of y. Consequently, we can find the x- and y-derivatives of $z = f(x, y)$ by holding the other variable constant and using formulas for derivatives of functions of one variable from earlier chapters.

Example 3 Find the x- and y-derivatives of $f(x, y) = x^3y - x^2y^5 + x$.

SOLUTION To obtain the x-derivative, we consider y to be a constant and differentiate with respect to x:

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}(x^3y - x^2y^5 + x) = \left[\frac{\partial}{\partial x}(x^3)\right]y - \left[\frac{\partial}{\partial x}(x^2)\right]y^5 + \frac{\partial}{\partial x}(x)\\ &= 3x^2y - 2xy^5 + 1.\end{aligned}$$

To find the y-derivative, we hold x fixed and differentiate with respect to y:

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(x^3y - x^2y^5 + x) = x^3\left[\frac{\partial}{\partial y}(y)\right] - x^2\left[\frac{\partial}{\partial y}(y^5)\right] + \frac{\partial}{\partial y}(x)\\ &= x^3 - 5x^2y^4. \ \square\end{aligned}$$

Example 4 What are $g_x(2, 5)$ and $g_y(2, 5)$ for $g(x, y) = x^2e^{3y}$?

SOLUTION Differentiating with respect to x with y constant gives

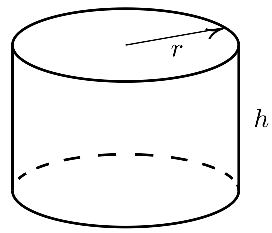
$$g_x(x, y) = \frac{\partial}{\partial x}(x^2e^{3y}) = 2xe^{3y}.$$

To differentiate with respect to y with x constant, we need the Chain Rule formula $\dfrac{d}{dy}(e^{f(y)}) = e^{f(y)}f'(y)$ for one variable rewritten using the partial derivative symbol $\dfrac{\partial}{\partial y}$. We obtain with $f(y) = 3y$

$$g_y(x, y) = \frac{\partial}{\partial y}(x^2e^{3y}) = x^2e^{3y}\frac{\partial}{\partial y}(3y) = 3x^2e^{3y}.$$

Setting $x = 2$ and $y = 5$ in these formulas gives $g_x(2, 5) = 2(2)e^{3(5)} = 4e^{15}$ and $g_y(2, 5) = 3(2)^2e^{3(5)} = 12e^{15}$. $\square$

Example 5 The volume of a right circular cylinder of radius r and height h is equal to the product $V(r,h) = \pi r^2 h$ of its height h and the area πr^2 of its base (Figure 3). What is the rate of change of the volume with respect to the radius and what is its geometric significance?



[Volume] $= \pi r^2 h$

[Lateral surface area] $= 2\pi r h$

FIGURE 3

SOLUTION The rate of change of V with respect to r is $\dfrac{\partial V}{\partial r} = \dfrac{\partial}{\partial r}(\pi r^2 h) = 2\pi r h$. It equals the area of the lateral surface (the sides) of the cylinder, which is given by the circumference of the base $2\pi r$ of the cylinder, multiplied by the height h. □

A geometric interpretation of partial derivatives

When we hold y equal to a constant $y = y_0$, $z = f(x,y)$ becomes the function $z = f(x, y_0)$ of x, whose graph is the intersection of the surface $z = f(x,y)$ with the vertical plane $y = y_0$ (Figure 4). The x-derivative $f_x(x_0, y_0)$ is the slope in the positive x-direction of the tangent line to this curve at $x = x_0$.

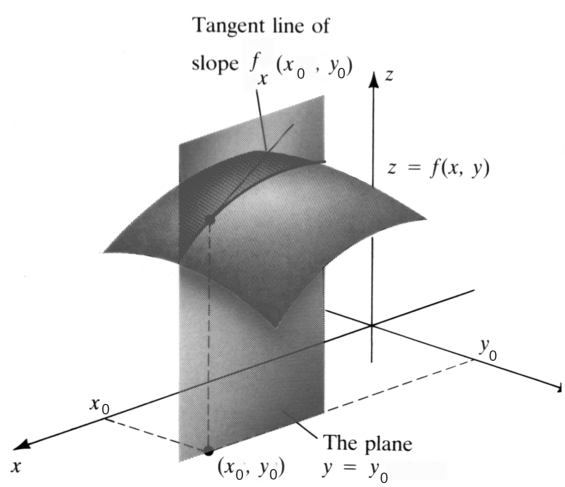


FIGURE 4

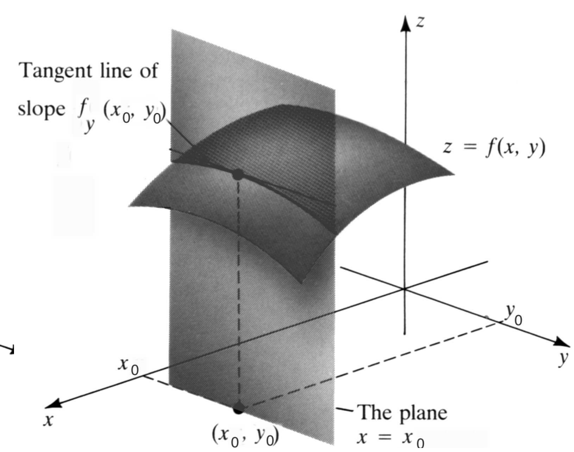


FIGURE 5

Similarly, when we hold x equal to a constant x_0, $z = f(x,y)$ becomes the function $z = f(x_0, y)$ of y, whose graph is the intersection of the surface with the plane $x = x_0$ (Figure 5), and the y-derivative $f_y(x_0, y_0)$ is the slope in the positive y-direction of the tangent line to this curve at $y = y_0$.

Example 6 The MONKEY SADDLE in Figure 6 is the graph of $g(x,y) = \frac{1}{3}y^3 - x^2y$. The curves drawn with heavy lines are the intersections of the surface with the planes $y = 1$ and $x = 2$. **(a)** What is the slope in the positive x-direction at $(2,1)$ of the intersection with $y = 1$? **(b)** What is the slope in the positive y-direction at $(2,1)$ of the intersection with $x = 2$?

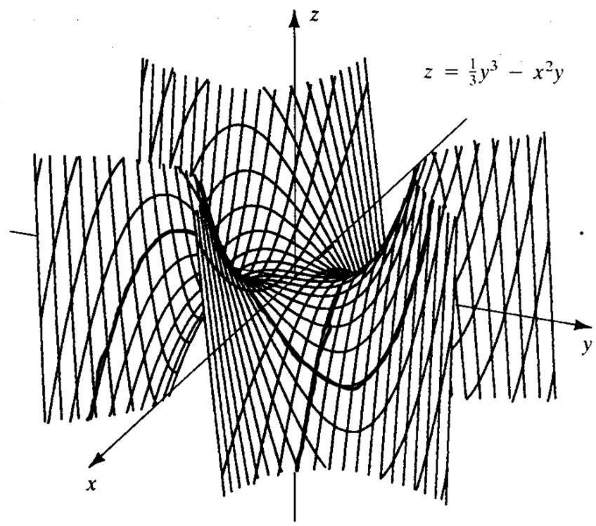


FIGURE 6

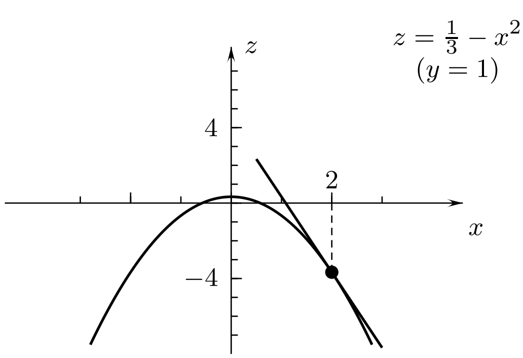
SOLUTION **(a)** The slope at $(2,1)$ of the intersection of the surface with the plane $y = 1$ is the partial derivative

$$\left.\frac{\partial g}{\partial x}\right|_{(2,1)} = \left[\frac{\partial}{\partial x}(\tfrac{1}{3}y^3 - x^2y)\right]_{(2,1)} = \Big[-2x\Big]_{(2,1)} = -2(2) = -4. \quad \textbf{(4)}$$

(b) The slope at $(2,1)$ of the intersection with the plane $x = 2$ is

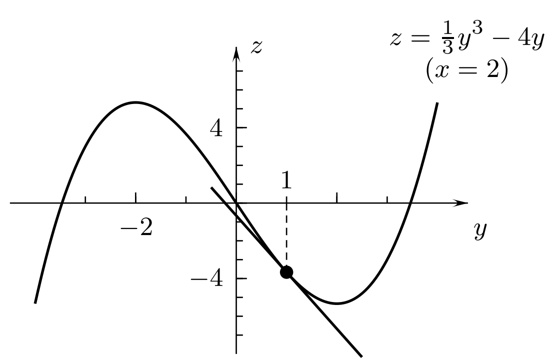
$$\left.\frac{\partial g}{\partial y}\right|_{(2,1)} = \left[\frac{\partial}{\partial y}(\tfrac{1}{3}y^3 - x^2y)\right]_{(2,1)} = \Big[y^2 - 4\Big]_{(2,1)} = 1^2 - 4 = -3. \;\square \quad \textbf{(5)}$$

When we set $y = 1$ in the equation $z = \frac{1}{3}y^3 - x^2y$, we obtain the equation $z = \frac{1}{3} - x^2$ for the cross section in terms of x and z. Figure 7 shows the graph of this equation and its tangent line at $x = 2$ in an xz-plane. The slope of the tangent line is the x-derivative **(4)** of $z = g(x,y)$ at $(2,1)$.



[Slope] $= g_x(2,1)$

FIGURE 7



[Slope] $= g_y(2,1)$

FIGURE 8

On the other hand, when we set $x = 2$ in the equation $z = \frac{1}{3}y^3 - x^2y$, we obtain the equation $z = \frac{1}{3}y^3 - 4y$ for this cross section in terms of x and z, whose graph is shown in the yz-plane of Figure 8 with its tangent line at $y = 1$. The slope of this tangent line is the y-derivative **(4)** of g at $(2, 1)$.

Estimating partial derivatives from tables

In the next example we estimate partial derivatives of a function of two variables whose values are given in a table by employing procedures that we used Section 2.5 to estimate derivatives of functions of one variable from tables.

Example 7 The table below is from a study of the effect of exercise on the blood pressure of women. $P = P(t, E)$ is the average blood pressure, measured in millimeters of mercury (mm Hg), of women of age t years who are exercising at the rate of E watts.[1] What is the approximate rate of change with respect to age of the average blood pressure of forty-five-year old women who are exercising at the rate of 100 watts?

TABLE 1. $P = P(t, E)$ (millimeters of mercury)

	$t = 25$	$t = 35$	$t = 45$	$t = 55$	$t = 65$
$E = 150$	178	180	197	209	195
$E = 100$	163	165	181	199	200
$E = 50$	145	149	167	177	181
$E = 0$	122	125	132	140	158

SOLUTION ONE ANSWER, USING A RIGHT DIFFERENCE QUOTIENT: The rate of change with respect to age of the average blood pressure of forty-five-year old women who are exercising at the rate of 100 watts is $P_t(45, 100)$. It is approximately equal to the average rate of change of $P(t, 100)$ with respect to t from $t = 45$ to $t = 55$:

$$P_t(45,100) \approx \frac{P(55,100) - P(45,100)}{55 - 45} = \frac{199 - 181}{10}$$
$$= 1.8 \text{ millimeters of mercury per year.}$$

ANOTHER ANSWER, USING A LEFT DIFFERENCE QUOTIENT: $P_t(45, 100)$ is approximately equal to the average rate of change of $P(t, 100)$ with respect to t rom $t = 35$ to $t = 45$:

$$P_t(45,100) \approx \frac{P(45,100) - P(35,100)}{45 - 35} = \frac{181 - 165}{10}$$
$$= 1.6 \text{ millimeters of mercury per year.}$$

A THIRD ANSWER, USING A CENTERED DIFFERENCE QUOTIENT: $P_t(45, 100)$ is approximately equal to the average rate of change of $P(t, 100)$ with respect to t from $t = 35$ to $t = 55$:

$$\frac{\partial P}{\partial t}(45,100) \approx \frac{P(55,100) - P(35,100)}{55 - 35} = \frac{199 - 165}{20}$$
$$= 1.7 \text{ millimeters of mercury per year.} \ \square$$

[1]Data adapted from *Geigy Scientific Tables*, edited by C. Lentner, Vol. 5, Basel, Switzerland: CIBA-GEIGY Limited, 1990, p. 29.

To estimate first derivatives at points that are between those in a table, we can use average rates of change with nearby points that are in the table, as in the next example.

Example 8 Based on the data in Table 1, what is the approximate rate of change of $P = P(t, E)$ with respect to E at $t = 62, E = 75$?

SOLUTION ONE ANSWER: If we use values at $t = 55$ with $E = 50$ and $E = 100$, we obtain

$$\left.\frac{\partial P}{\partial E}\right|_{(62,75)} \approx \frac{P(55,100) - P(55,50)}{100 - 50} = \frac{199 - 177}{50}$$
$$= 0.44 \text{ millimeters of mercury per watt.}$$

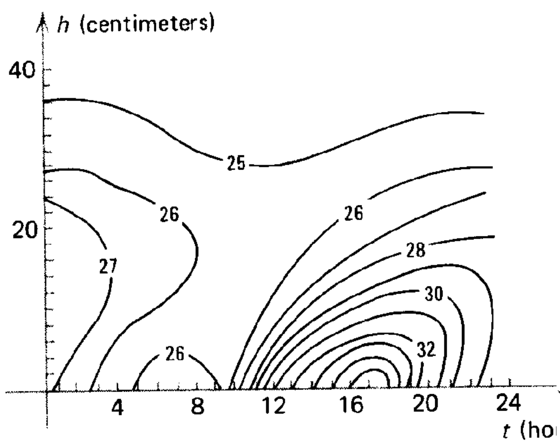
ANOTHER ANSWER: The values at $t = 65$ with $E = 50$ and $E = 100$ yield

$$\left.\frac{\partial P}{\partial E}\right|_{(62,75)} \approx \frac{P(65,100) - P(65,50)}{100 - 50} = \frac{200 - 181}{50}$$
$$= 0.38 \text{ millimeters of mercury per watt.} \square$$

Estimating partial derivatives from level curves

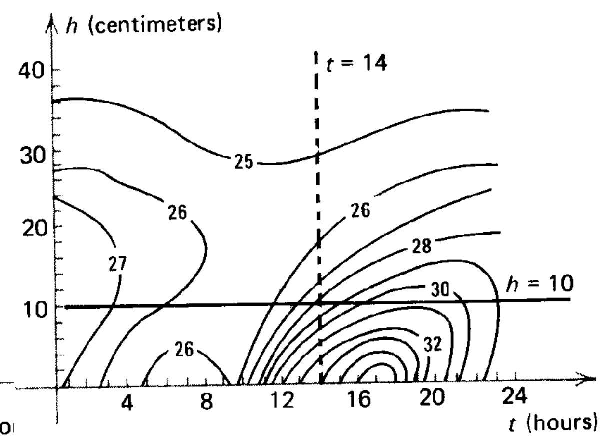
We can estimate first-order partial derivatives of a function from a drawing of its level curves by using average rates of change with values at points on the level curves.

Example 9 Figure 9 shows level curves of the temperature $T = T(t, h)$ (degrees Fahrenheit) as a function of time t (hours) and the depth h (centimeters) beneath the surface of the ground at O'Neil, Nebraska, from midnight one day ($t = 0$) until midnight the next.[2] What is the approximate rate of change of the temperature with respect to time at 2:00 PM at a point ten centimeters beneath the surface of the ground?



Level curves of $T = T(t, h)$

FIGURE 9



The line $h = 10$

FIGURE 10

[2]Data adapted from *Fundamentals of Air Pollution* by S. Williamson, Reading, MA: Addison Wesley, 1973, p. 162.

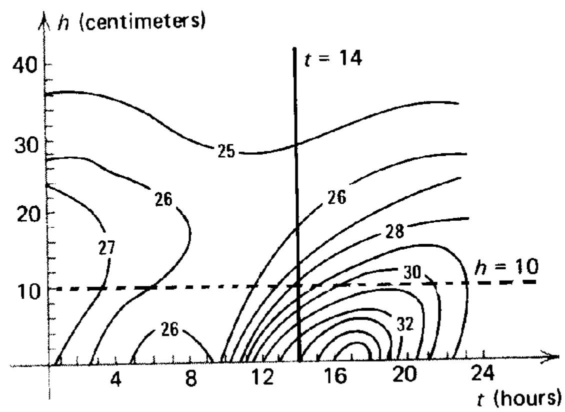
SOLUTION Because $t = 14$ at 2:00 PM and $h = 10$ ten centimeters below the surface of the ground, the required rate of change is the t-derivative $T_t(t, 10)$ at $t = 14$. To find its approximate value, we draw the horizontal line $h = 10$, as in Figure 10. The point $(14, 10)$ is between the level curves $T = 28$ and $T = 29$ of the temperature, so the change ΔT in temperature from the left curve to the right curve is 1 degree. The horizontal distance Δt along $h = 10$ from the left curve to the right curve is approximately 1 hour. Consequently,

$$T_t(14, 10) \approx \frac{\Delta T}{\Delta t} \approx \frac{1 \text{ degrees}}{1 \text{ hour}} = 1 \text{ degree per hour.} \square$$

Example 10 What is the approximate rate of change of the temperature with respect to depth at 2:00 PM at a point ten centimeters beneath the surface of the ground?

SOLUTION Along the vertical line $t = 14$ in Figure 11, the distance between the level curves $T = 28$ above and $T = 29$ below the point $h = 10, t = 14$ is approximately 2 centimeters, measured on the h-axis. The temperature changes $\Delta T = -1$ degree as h increases $\Delta h = 2$ centimeters, so that

$$T_h(14, 10) \approx \frac{\Delta T}{\Delta h} \approx \frac{-1 \text{ degree}}{2 \text{ centimeters}} = -\tfrac{1}{2} \text{ degree per centimeter.} \square$$



The line $t = 14$

FIGURE 11

Interactive Examples 14.3

Interactive solutions are on the web page http//www.math.ucsd.edu/˜ashenk/.†

1. What is the limit $\displaystyle\lim_{(x,y)\to(3,0)} \frac{x^2}{\cos(\sqrt{y})}$?

2. Find the partial derivatives **(a)** $\dfrac{\partial}{\partial x}(xy^5 - 4y^2 + 6x^4y^7)$ and **(b)** $\dfrac{\partial}{\partial y}(xy^5 - 4y^2 + 6x^4y^7)$.

3. What are W_x and W_y for $W(x, y) = \ln(1 - xy)$?

† In the published text the interactive solutions of these examples will be on an accompanying CD disk which can be run by any computer browser without using an internet connection.

4. The following table gives values of the air pressure $z = P(x, y)$ (millibars), measured x miles east and y miles north of New Orleans at one time on August 29, 2005 during hurricane Katrina.[3] Estimate the rate of change of the pressure with respect to x and y in New Orleans at that time.

	$x = -30$	$x = 0$	$x = 30$
$y = 30$	997	990	950
$y = 0$	985	977	960
$y = -30$	968	950	990

5. Figure 12 shows level curves of the temperature $z = T(x, y)^\circ$F in a square plate. Find approximate values of **(a)** $T_x(3, 2)$ and **(b)** $T_x(3, 2)$.

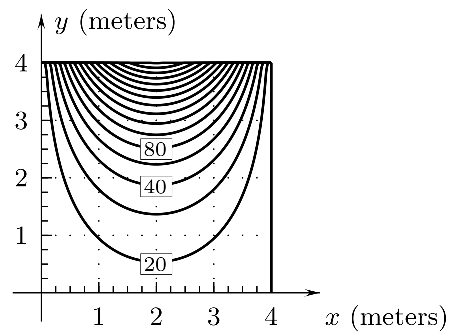


FIGURE 12

Exercises 14.3

AAnswer provided. **O**Outline of solution provided. **C**Graphing calculator or computer required.

CONCEPTS:

1. What is the h-derivative of the volume $V = \pi r^2 h$ from Example 5 and what is the geometric interpretation of this formula?

2. The result of Example 5 has the following interpretation: If the radius of the cylinder is increased slightly without changing its height, then the increase in the volume is approximately equal to the lateral surface area of the cylinder multiplied by the change in the radius. **(a)** Give a similar interpretation of Exercise 1. **(b)** Why is the term "approximately" not needed in your answer to part (a)?

3. **(a)** What is the x-derivative of $A(x, y) = xy^3$ at $(5, 2)$? **(b)** Describe the intersection of the graph $z = xy^3$ with the vertical plane $y = 2$. **(c)** How are the results of parts (a) and (b) related?

4. Why is the surface $z = \frac{1}{3}y^3 - x^2 y$ in Figure 6 called a "monkey saddle"?

[3] Data adapted from "Tropical Cyclone Report, Hurricane Katrina", National Hurricane Center, December, 2005.

BASICS:

Find the limits in Problems 5 through 6.

5.[O] $\displaystyle\lim_{(x,y)\to(-5,3)} \frac{x^2y}{x^2-y}$.

6.[A] $\displaystyle\lim_{(x,y)\to(-2,7)} x\sin(xy)$

7. $\displaystyle\lim_{(x,y)\to(0,0)} e^{3x-2y}$.

8. $\displaystyle\lim_{(x,y)\to(1,3)} \ln(x^2+3y)$.

9.[O] $\displaystyle\lim_{(x,y)\to(3,2)} \frac{\sin(x-y)}{1+x^2+y^2}$

10. $\displaystyle\lim_{(x,y)\to(0,0)} \left(5-\frac{1}{x^2}-\frac{1}{y^2}\right)$

Find the derivatives in Problems 11 through 19.

11.[O] $\dfrac{\partial}{\partial x}(x^3y^2-x+y)$

12.[A] $\dfrac{\partial}{\partial y}(x^2e^{3y}+y^2e^{3x})$

13. $\dfrac{\partial}{\partial x}(xe^y+6x^2-y)$

14.[O] F_x and F_y for $F(x,y)=\sin(x^2y^4)$

15.[O] $\partial W/\partial x$ and $\partial W/\partial y$ for $G(x,y)=\sin(x^2+y)$.

16. H_x and H_y for $H(x,y)=(x^2+x+1)(y^2+y-3)$

17. The y-derivative of $G(x,y)=x^2\sin(xy)+y-x$

18.[A] The first derivatives of $P(u,v)$ $=e^{u^2}\cos(v^2)$

19. The first derivatives of $Q(x,y)=x^{1/2}y^{1/4}+x^2y^4$.

20.[A] The volume of a right circular cone of height h meters and with a base of radius r meters is $V=\frac{1}{3}\pi r^2h$ cubic meters. What is the rate of change of the volume with respect to the radius?

21. If a constant current of I amperes flows through a circuit with a resistance of 100 ohms for t seconds, it will produce $H(I,t)=23.9I^2t$ calories of heat.[4] What are the rates of change of the heat production **(a)** with respect to I and **(b)** with respect to t at $I=10, t=5$? Give the units.

22.[O] Use the following table of values of $z=g(x,y)$ to estimate **(a)** $g_x(2,5)$ and **(b)** $g_y(2,5)$.

	$x=1$	$x=1.5$	$x=2$	$x=2.5$	$x=3$
$y=5.2$	150	160	172	184	195
$y=5.0$	187	200	212	223	235
$y=4.8$	231	242	253	266	278
$y=4.6$	273	283	293	305	316

[4] Data adapted from *CRS Handbook of Chemistry and Physics*, 62nd edition, Boca Raton, FL: CRC Press, Inc., 1981, p. F-98.

23.[A] The table below gives the volume $V = V(p,T)$ (cubic feet) of a pound of a p-percent solution of sulfuric acid in water that is at a temperature of T°C.[5] **(a)** Does the volume of a pound of a solution increase or decrease as the temperature increases? **(b)** Suppose that two solutions are at the same temperature but one contains a greater concentration of sulfuric acid. Which has the greater volume? **(c)** What rate of change is represented by $V_p(15, 80)$ and what is its approximate value? **(d)** What is the approximate value of $V_T(p,T)$ for $10 \le p \le 20$ and $60 \le T \le 100$?

	$p = 10\%$	$p = 15\%$	$p = 20\%$	$p = 25\%$
$T = 100^\circ$C	0.0157	0.0152	0.0147	0.0143
$T = 80^\circ$C	0.0155	0.0150	0.0145	0.0141
$T = 60^\circ$C	0.0153	0.0148	0.0143	0.0139
$T = 40^\circ$C	0.0151	0.0146	0.0142	0.0138

24.[A] The next table gives the wind chill $W = W(T,v)$ (degrees Fahrenheit) as a function of the Fahrenheit temperature T and the velocity of the wind v, measured in miles per hour for five temperatures below the freezing point of water (32°F) and four wind speeds. $W(T,v)$ is the temperature which with no wind has the same cooling effect as temperature T°F in a wind of velocity v miles per hour. **(a)** Is $W = W(T,v)$ an increasing or a decreasing function of T for fixed v? **(b)** What is the approximate rate of change of $W = W(T,v)$ with respect to T if the temperature is 0°F and the wind velocity is 20 miles per hour? **(c)** Is wind chill an increasing or a decreasing function of velocity for fixed temperature? **(d)** What is the approximate rate of change of the wind chill with respect to the velocity of the wind when the temperature is 0°F and the wind velocity is 20 miles per hour?

	$T = -20$	$T = -10$	$T = 0$	$T = 10$	$T = 20$
$v = 30$	-79	-64	-49	-33	-18
$v = 20$	-67	-53	-39	-24	-10
$v = 10$	-46	-34	-22	-9	3
$v = 0$	-20	-10	0	10	20

[5] Data adapted from *Handbook of Engineering Materials* by F. Miner and J. Seastone, New York, NY: John Wiley & Sons, 1955, p. 3-407

25. The next table gives the amount of food $F = F(w,t)$ (pounds) required each day by a horse that weighs w pounds and is ridden t hours a day. **(a)** Give approximate values of $F_w(1000,4)$ and $F_t(1000,4)$ with units. **(b)** What is it about horses causes $F = F(w,t)$ to be an increasing function of w for fixed t and an increasing function of t for fixed w?

	$w = 800$	$w = 900$	$w = 1000$	$w = 1100$	$w = 1200$
$t = 6$	18.7	20.5	22.2	23.8	25.4
$t = 4$	17.9	19.5	21.2	22.8	24.3
$t = 2$	16.9	18.5	20.1	21.5	23.0
$t = 0$	12.9	14.1	15.3	16.4	17.5

26.[O] Use the level curves of $z = G(x,y)$ in Figure 13 to find approximate values of **(a)** $G_x(3,3)$ and **(b)** $G_y(3,3)$.

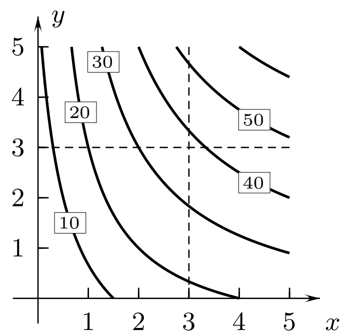
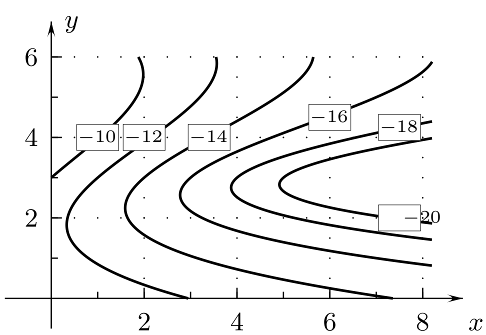


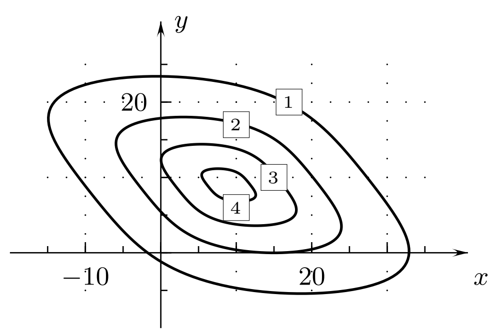
FIGURE 13

27.[O] Use the level curves of $z = K(x,y)$ in Figure 14 to give its approximate x- and y-derivatives at $(6,2)$.



Level curves of $K(x,y)$

FIGURE 14

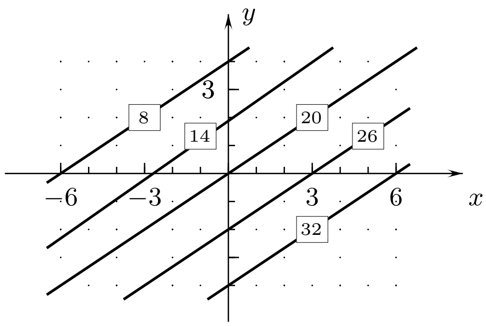


Level curves of $L(x,y)$

FIGURE 15

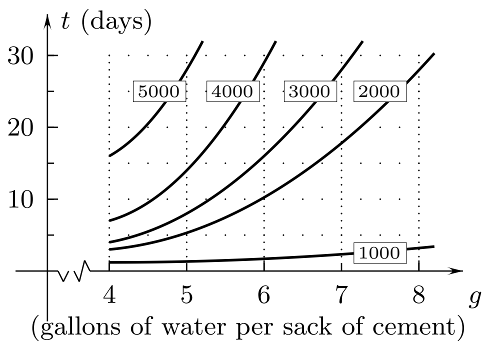
28.[A] Based on the level curves of $z = L(x,y)$ in Figure 15, what is the approximate value of $L_x(20,10)$?

29. The x- and y-derivatives of the function $z = h(x, y)$ of Figure 16 are constant. What are their values?



Level curves of $z = h(x, y)$

FIGURE 16



Level curves of $S = S(g, t)$

FIGURE 17

30. Figure 17 gives level curves of the compressive strength $S = S(g, t)$ (pounds per square inch) of portland concrete that is made with g gallons of water per sack of cement and that has cured t days.[6] What are the approximate values of $S_g(6, 15)$ and $S_t(6, 15)$?

EXPLORATION:

31.[O] Figure 18 shows the graph of the function $z = P(x, 2)$ of x that is obtained from $z = P(x, y)$ by setting $y = 2$, and Figure 16 shows the graph of the function $z = P(3, y)$ of y that is obtained from $z = P(x, y)$ by setting $x = 3$. Use the graphs to find the approximate values of $P_x(3, 2)$ and $P_y(3, 2)$.

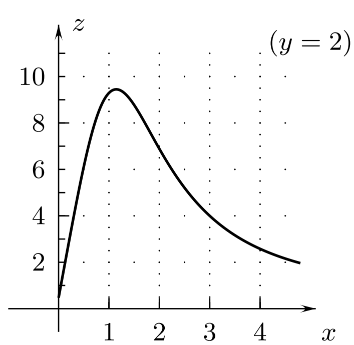


FIGURE 18

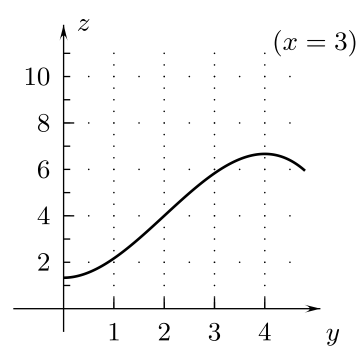


FIGURE 19

32.[A] **(a)** Draw in an xz-plane the intersection of the plane $y = 2$ with the graph of $H(x, y) = \frac{1}{2}y^2 - x^2$ and the tangent line to the curve whose slope in the positive x-direction is $H_x(1, 2)$. **(b)** Draw in a yz-plane the intersection of the plane $x = 1$ with the graph of $H(x, y)$ and the tangent line to the curve whose slope in the positive y-direction is $H_y(1, 2)$.

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[6] Data adapted from *Handbook Of Engineering Materials*, Ibid., p. 4-14

33. Figure 20 gives level curves of the amount of solar radiation $R = R(t, L)$ (calories per square centimeter) during a cloudless day at a latitude of L (degrees) and at time t (month) of the year. What are the approximate values of **(a)** R, **(b)** $\partial R/\partial t$, and **(c)** of $\partial R/\partial L$ at May 1 and a latitude of 40°. **(d)** Why, based on the seasons, is $R(6, 0)$ greater than $R(6, -60)$ and $R(1, 60)$? **(d)** Why are $R(1, 80)$ and $R(6, -80)$ zero? **(e)** Where and when is the solar radiation the greatest?

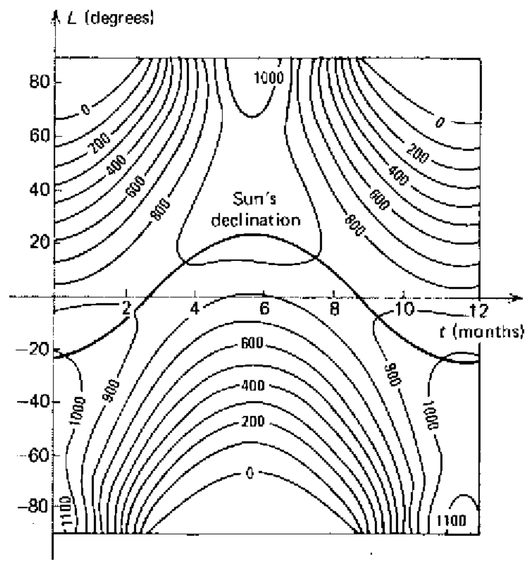

FIGURE 20

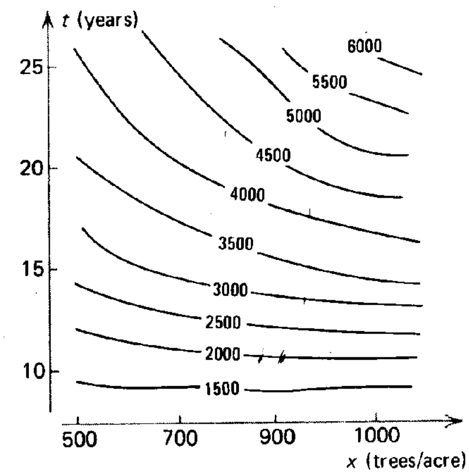

FIGURE 21

34. Figure 21 shows level curves of the amount of the yield $Y = Y(x, t)$ (cubic feet per acre) from a pine plantation with x trees per acre that are harvested t years after planting. **(a)** Determine without doing any calculations whether $Y_t(700, 20)$ is less than or greater than $Y_t(1000, 15)$. Explain your reasoning and describe what this indicates about the trees. **(b)** Is it better to have 600 trees per acre or 1000 trees per acre if your only goal is to maximize the yield?

35. The FETCH of the wind at a point on a body of water is the distance that the wind has blown over water before it reaches the point. The next table gives the height $h = h(v, f)$ (feet) of waves as a function of the velocity v (knots) and of the fetch f (nautical miles). (Knots are nautical miles per hour). **(a)** Based on the table, what is $h_f(10, f)$ for all f? What does this say about the waves? **(b)** Based on the table, is $h_v(40, f)$ an increasing or a decreasing function of f? What does this say about the waves? **(c)** What do you think would happen to $h(v, f)$ as $f \to \infty$ for each fixed v and why?

	$v = 10$	$v = 20$	$v = 30$	$v = 40$
$f = 1000$	3	8	18	50
$f = 500$	3	8	18	47.5
$f = 200$	3	7.5	17	39.5
$f = 100$	3	7	14.5	31
$f = 50$	3	6	12	22

36.[A] Use polar coordinates to find the following limits or show that they do not exist:
(a) $\lim_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+y^2}}$, **(b)** $\lim_{(x,y)\to(0,0)} \dfrac{x+y}{x^2+y^2}$, **(c)** $\lim_{(x,y)\to(0,0)} \dfrac{(x^2+y^2)^2+x^3y^3}{(x^2+y^2)^2}$.

37. Use polar coordinates to find the value of $\lim_{(x,y)\to(0,0)} \dfrac{y}{x}$ or to show that the limit does not exist.

38. Show that $\lim_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4+y^2}$ does not exist by considering points (x,y) that approach $(0,0)$ along different parabolas.

39. The total area of the base and lateral surface of a right circular cone of height h and with base of radius r is $A(r,h) = \pi r^2 + \pi r\sqrt{r^2+h^2}$ for positive h and r. **(a)** What is the limit of $A(h,r)$ as $h \to 0^+$ for fixed $r > 0$? **(b)** What is the limit of $\dfrac{A(h,r)}{\pi r^2}$ as $r \to \infty$ for fixed $hr > 0$? Give geometric interpretations of the results in parts (a) and (b).

40. If a gas has density ρ_0 grams per cubic centimeter at 0°C and pressure of one atmosphere, then its density at T°C and pressure P atmospheres is $\rho(T,P) = \dfrac{\rho_0 P}{1+\frac{1}{273}T}$ grams per cubic centimeter.[7]
(a) Find formulas for $\partial\rho/\partial T$ and $\partial\rho/\partial P$ in terms of T, P, and the parameter ρ_0. Give the units. **(b)** One of the derivatives in part (a) is positive and the other is negative for $T > -273$ and positive P. What properties of gasses do these illustrate.

41. FInd the approximate maximum and minimum values of $W_y(x,y)$ for $0 \le x \le 5, 1 \le y \le 4$, where $z = W(x,y)$ is the the function whose level curves are shown in Figure 22.

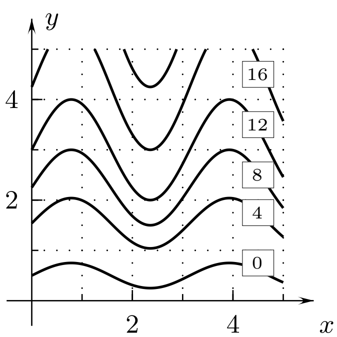

FIGURE 22

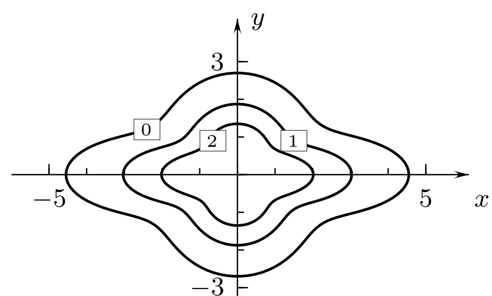

FIGURE 23

42. **(a)** Describe the shape of the graph of the function $Z = Z(x,y)$ whose level curves are shown in Figure 23. **(b)** Aore $Z_x(0,y)$ and $Z_y(x,0)$ even or odd functions of y and x?

C **43.** **(a)** Find $A(x)$ and $B(y)$ such that the intersection of the graph of $f(x,y) = x^2e^{(1-y^2)/2}$ with the plane $y = -1$ has the equations $z = A(x), y = -1$, and its intersection with the plane $x = 2$ has the equations $z = B(y), x = 2$. **(b)** Draw the graph of $z = A(x)$ in an xz-plane with its tangent line whose slope is $f_x(2,-1)$. **(c)** Draw the graph of $z = B(y)$ in an yz-plane with its tangent line whose slope is $f_y(2,-1i)$.

C **44.** **(a)** Generate the intersection of the graph of $g(x,y) = x^4 - xy^2 + y^3$ with the plane $y = -1$ and the tangent line whose slope is $g_x(1,-1)$. **(b)** Generate the intersection of the graph of g with the plane $x = 1$ and the tangent line whose slope is $g_y(1,-1)$.

[7] Ibid, p. F-94

45. The volume of water (liters) in the body of a person who weighs w kilograms and is h centimeters high can be predicted with the formula $V(w,h) = 0.135w^{2/3}h^{1/2}$.[8] Suppose that a man is 169 centimeters high and that at his current weight, the volume of water in his body would increase by approximately $0.03\Delta w$ if his weight increased by a small amount Δw. Based on the formula for V, how much does he weigh?

(End of Section 14.3)

[8]Data adapted from *Report of the Task Group on Reference Man*, International Commission on Radiological Protection, TarryTown, NY: Elsevier Science, Inc., 1975, p. 28.