Section 14.8

(3/24/08)

Functions of more than three variables

Overview: *In this section we discuss functions with more than three variables and their derivatives. The necessary definitions, results, and procedures are analogous to those for functions with two or three variables.*

Topics:

- **n-*dimensional Euclidean space and functions of* n *variables***
- ***Partial derivatives***
- ***Graphs of functions and their tangent planes***
- ***Gradient vectors and directional derivatives***
- ***Level surfaces and their tangent planes***

n-*dimensional Euclidean space and functions of* n *variables*

Most results concerning derivatives of functions of two or three variables carry over to functions with more variables when we provide for the extra variables in definitions, theorems, and descriptions of procedures. The main difference is that we cannot visualize geometric interpretations of results and procedures when there are more than three variables. Nevertheless, we can use our geometric intuition to guide us in dealing with the higher dimensional cases by visualizing the analagous calculations and relationships in two- or three-dimensional space. For this purpose, we use terms such as "point," "plane," "surface," and "space" in discussions of the corresponding mathematical entities in dimensions higher than three.

A point in n-dimensional space, for an arbitrary positive integer n, is an ordered set $P = (x_1, x_2, x_3, \ldots, x_n)$ of n real numbers. The numbers are the coordinates of P. The set of all such points is called n-dimensional Euclidean space and is denoted $\Re^n$. $\Re^1$ is a coordinate line and $\Re^2$ is a coordinate plane.

A real-valued function f of n variables is defined on a subset of $\Re^n$:

Defintion 1 (Functions of* n *variables) *A function f of n variables, for any positive integer n, is a rule that assigns a number, denoted $f(x_1, x_2, x_3, \ldots, x_n)$, to each point in the domain of the function. The domain is a subset or all of $\Re^n$.*

Example 1 On July 21, 2006, Ford Motor stock sold for \$6.19 per share, Hewlett Packard stock sold for \$31,80 per share, Motorola stock sold for \$20.60 per share, Pepsi Cola stock sold for \$62.48 per share, and Yahoo stock sold for \$25.27 per share. Give a formula for the cost $C(x_1, x_2, x_3, x_4, x_5)$ on that day of x_1 shares of Ford Motor stock, x_2 shares of Hewlett Packard stock, x_3 shares of Motorola stock, x_4 shares of Pepsi Cola stock, and x_5 shares of Yahoo stock. What is the domain of this function?

Solution $C(x_1, x_2, x_3, x_4, x_5) = 6.19x_1 + 31.80x_2 + 20.60x_3 + 62.48x_4 + 25.27x_5$ dollars. Since each of the variables can be zero or a positive integer, the domain is the subset of $\Re^5$ consisting of the points $(x_1, x_2, x_3, x_4, x_5)$, where x_1, x_2, x_3, x_4, and x_5 are nonnegative integers. □

If a function of n variables is given by a with no extra restrictions on the variables, we assume that its domain consists of all points in $\Re^n$ where the formula is defined.

Example 2 What is the domain of the function $g(x, y, z, w) = \sqrt{x + y + z + w}$?

Solution Since $\sqrt{t}$ is defined for $t \geq 0$, the domain of g is the subset $\{(x, y, z, w) : x+y+z+w \geq 0\}$ of four-dimensional $xyzw$-space. □

Partial derivatives,

A partial derivative $f_{x_j} = \partial f/\partial x_j$ of a function f with n variables $(x_1, x_2, \ldots, x_n)$ for any $n > 3$, like a partial derivative with two or three variables, is obtained by differentiating with respect to x_j while holding the other variables constant.

Example 3 Find the derivative P_{xz} of $P(x, y, z, v, w) = xe^y \sin z \cos v \ln w$.

SOLUTION The x-derivative is $P_x = \dfrac{\partial}{\partial x}(xe^y \sin z \cos v \ln w) = e^y \sin z \cos v \ln w$. The xz-derivative is $P_{xz} = \dfrac{\partial}{\partial z}(P_x) = \dfrac{\partial}{\partial z} = \dfrac{\partial}{\partial z}(e^y \sin z \cos v \ln w) = e^y \cos z \cos v \ln w$. □

Example 4 A steeple consists of a rectangular box of width x, depth y, and height y with a rectangular pyramid shape of height h on top of it, as in Figure 1. The volume of the steeple is $V(x, y, z, w) = xyz + \frac{1}{3}xyh$. What are the first-order partial derivatives of the volume?

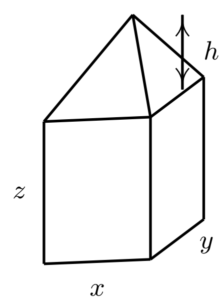

FIGURE 1

SOLUTION The x-derivative is $V_x = \dfrac{\partial}{\partial x}(xyz + \frac{1}{3}xyh) = yz + \frac{1}{3}yh$. The y-derivative is $V_y = \dfrac{\partial}{\partial y}(xyz + \frac{1}{3}xyh) = xz + \frac{1}{3}xh$. The z-derivative is $V_z = \dfrac{\partial}{\partial z}(xyz + \frac{1}{3}xyh) = xy$. The h-derivative is $V_h = \dfrac{\partial}{\partial h}(xyz + \frac{1}{3}xyh) = \frac{1}{3}xy$. □

Graphs of functions and their tangent planes

The definitions of graphs and level surfaces of functions with two variables generalize to functions of three or more variables as follows.

Definition 2 (Graphs with n variables) *The* GRAPH *of a function f with the n variables $(x_1, x_2, x_3, \ldots, x_n)$ is the surface $x_{n+1} = f(x_1, x_2, x_3, \ldots, x_n)$ in $\Re^{n+1}$.*

We cannot draw or visualize graphs of functions with three or more variables because they are surfaces in $\Re^n$ with $n \geq 4$.

The definition of tangent planes to graphs of functions with three or more variables is obtained by adding variables to the definition for two variables from Section 14.5.

Definition 3 *The tangent plane to the graph $x_{n+1} = f(x_1, x_2, \ldots, x_n)$ at $(a_1, a_2, \ldots, a_n)$ is*

$$x_{n+1} = f + f_{x_1}(x_1 - a_1) + f_{x_2}(x_2 - a_2) + \cdots + f_{x_n}(x_n - a_n) \tag{1}$$

with the function and its derivatives evaluated at $(a_1, a_2, \ldots, a_n)$.

Example 5 Give an equation of the tangent plane to the graph of $w = x^2 + y^2 + z^2$ at $x = 1, y = 2, z = 3$.

SOLUTION The value of the function at $x = 1, y = 2, z = 3$ is $w(1, 2, 3) = 1^2 + 2^2 + 3^2 = 14$. The first-order partial derivatives are $w_x = \frac{\partial}{\partial x}(x^2 + y^2 + z^2) = 2x$, $w_y = \frac{\partial}{\partial y}(x^2 + y^2 + z^2) = 2y$, and $w_z = \frac{\partial}{\partial x}(x^2 + y^2 + z^2) = 2z$. Their values at $(1, 2, 3)$ are $w_x(1, 2, 3) = 2(1) = 1, w_y(1, 2, 3) = 2(2) = 2$, and $w_z(1, 2, 3) = 2(3) = 6$. Consequently, the tangent plane is $w = 14 + 2(x - 1) + 4(y - 2) + 6(z - 3)$.□

Gradient vectors and directional derivatives

A vector in $\Re^n$ for $n > 3$ is an ordered n-tuple of numbers $\mathbf{A} = \langle a_1, a_2, \ldots, a_n\rangle$. Vectors in $\Re^n$ are added component by component and to multiply a vector by a number, each component is multiplied by the number:

$$\langle a_1, a_2, \ldots, a_n\rangle + \langle b_1, b_2, \ldots, b_n\rangle = \langle a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n\rangle$$
$$k\langle a_1, a_2, \ldots, a_n\rangle = \langle ka_1, ka_2, \ldots, ka_n\rangle.$$

The displacement vector $\overrightarrow{PQ}$ from a point $P = (x_1, x_2, \ldots, x_n)$ to a point $Q = (y_1, y_2, \ldots y_n)$ is obtained by subtracting the coordinates of P from the coordinates of Q:

$$\overrightarrow{PQ} = \langle y_1 - x_1, y_2 - x_2, \ldots, y_n - x_n\rangle.$$

The length of a vector is the square root of the sum of the squares of its components,

$$|\langle a_1, a_2, \ldots, a_n\rangle| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

and a unit vector is a vector of length 1.

Two nonzero vectors are said to have the same direction if one equals a positive number multiplied by the other. The unit vector with the same direction as a nonzero vector can be obtained by dividing the vector by its length.

The dot product of two vectors is the sum of the products of their components:

$$\langle a_1, a_2, \ldots, a_n\rangle \cdot \langle b_1, b_2, \ldots, b_n\rangle = a_1b_1 + a_2b_2 + \cdots + a_nb_n.$$

One vector is said to be perpendicular to another if their dot product is zero.

With this notation, the definitions of gradients and directional derivatives for $n > 3$ differ from those in the cases of $n = 2$ and 3 only in the numbers of coordinates and components:

Definition 4 (Gradient vectors and directional derivatives) **(a)** *The gradient vector of* $x_{n+1} = f(x_1, x_2, \ldots, x_n)$ *at* $(a_1, a_2, \ldots, a_n)$ *is*

$$\nabla f(a_1, a_2, \ldots, a_n) = \langle f_{x_1}, f_{x_2}, \ldots, f_{x_n}\rangle$$

with the partial derivatives evaluated at $(a_1, a_2, \ldots, a_n)$.

(b) *The directional derivative* $D_{\mathbf{u}}f$ *of* $z = f(x, y, z)$ *at* $(a_1, a_2, \ldots, a_n)$ *in the direction of the unit vector* $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ *is the* t*-derivative of the cross section* $x_{n+1} = f(a_1 + tu_1, a_2 + tu_2, \ldots, a_n + tu_n)$ *at* $t = 0$.

The procedure for finding a directional derivative and the basic results concerning gradient vectors in in $\Re^n$ for $n > 3$ are basically the same as in the cases of $n = 2$ and $n = 3$:

Theorem 1 **(a)** *The directional derivative* $D_u f(a_1, a_2, \ldots, a_n)$ *of* $x_{n+1} = f(x_1, x_2, \ldots, x_n)$ *is equal to the dot product of the gradient of* f *at* $(a_1, a_2, \ldots, a_n)$ *with the unit vector* $\mathbf{u}$*:*

$$\begin{aligned} D_u f(a_1, a_2, \ldots, a_n) &= \nabla f(a_1, a_2, \ldots, a_n) \cdot \mathbf{u} \\ &= f_{x_1} u_1 + f_{x_2} u_2 + \cdots + f_{x_n} u_n. \end{aligned} \tag{2}$$

The derivatives in these formulas are evaluated at $(a_1, a_2, \ldots, a_n)$*.*

(b) *The greatest directional derivative of* f *at* $(a_1, a_2, \ldots, a_n)$ *is the length* $|\nabla f|$ *of the gradient vector at that point, and the least directional derivative is the negative of the length of the gradient vector.*

(c) *If the gradient vector is not zero at* $(a_1, a_2, \ldots, a_n)$*, then the greatest directional derivative at the point is in the direction of the gradient, the least directional derivative is in the direction opposite to that of the gradient, and the directional derivative is zero in the directions perpendicular to the gradient.*

This theorem can be established by modifying proofs from Section 14.5 to deal with n variables for any $n \geq 2$ rather than with two variables.

Example 6 **(a)** What is the gradient vector of $w = x^2 + y^3 + z^4 + v^5$ at $(1, 1, 1, 1)$? **(b)** What is the derivative of $w = x^2 + y^2 + z^2 + w^2$ at $(1, 1, 1, 1)$ in the direction toward the origin? **(c)** What is the maximum directional derivative of $w = x^2 + y^2 + z^2 + w^2$ at $(1, 1, 1, 1)$?

SOLUTION **(a)** The gradient at (x, y, z, v) is $\nabla w = \langle w_x, w_y, w_z, w_v \rangle = \langle 2x, 3y^2, 4z^3, 5v^4 \rangle$ so $\nabla f(1, 1, 1, 1) = \langle 2, 3, 4, 5 \rangle$.

(b) The displacement vector from $P = (1, 1, 1, 1)$ to the origin $O = (0, 0, 0, 0)$ is $\overrightarrow{PO} = \langle 0-1, 0-1, 0-1, 0-1 \rangle = \langle -1, -1, -1, -1 \rangle$. Its length is $|\langle -1, -1, -1, -1 \rangle| = \sqrt{1^2 + 1^2 + 1^2 + 1^2} = \sqrt{4} = 2$. Consequently, the unit vector with that direction is $\mathbf{u} = \langle \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \rangle$.

By formula **(2)**, the directional derivative is

$$\begin{aligned} \nabla w \cdot \mathbf{u} &= \langle 2, 3, 4, 5 \rangle \cdot \langle \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2} \rangle \\ &= 2(\tfrac{1}{2}) + 3(\tfrac{1}{2}) + 4(\tfrac{1}{2}) + 5(\tfrac{1}{2}) = 14(\tfrac{1}{2}) = 7. \ \Box \end{aligned}$$

(c) By part (c) of Theorem 1, the maximum directional derivative of the function at $(1, 1, 1, 1)$ is equal to the length of the gradient vector, which is $|\langle 2, 3, 4, 5 \rangle| = \sqrt{2^2 + 3^2 + 4^2 + 5^2} = \sqrt{54}$. $\Box$

Level surfaces and their tangent planes

Level surfaces of functions with n-variables and their tangent planes are in $\Re^n$.

Definition 5 (Level surfaces and their tangent planes) **(a)** *A level surface of a function* $x_{n+1} = f(x_1, x_2, \ldots, x_n)$ *is the locus of points in* $\Re^n$ *that satisfy the equation* $f(x_1, x_2, \ldots, x_n) = c$ *with a constant* c*.*

(b) *If the gradient of* f *is not zero at a point* $(a_1, a_2, \ldots, a_n)$*, then the level surface through that point has a tangent plane there with the equation,*

$$f_{x_1}(x_1 - a_1) + f_{x_2}(x_2 - a_2) + \cdots + f_{x_n}(x_n - a_n) = 0.$$

The partial derivatives here are evaluated at $(a_1, a_2, \ldots, a_n)$*.*

Example 7 Give an equation for the tangent plane to the level surface $f = 5$ of $f(x, y, z, u, v) = e^x + e^y + e^u + e^v$ at the origin $(0, 0, 0, 0, 0)$ in $xyzuv$-space.

SOLUTION Since the derivatives $f_x = e^x, f_y = e^y, f_z = e^z, f_u = e^u$, and $f_v = e^v$ all have the value 1 at $(0, 0, 0, 0, 0)$, the tangent plane has the equation,

$$(x_1 - 0) + (x_2 - 0) + (x_3 - 0) + (x_4 - 0) + (x_5 - 0) = 0$$

which simplifies to $x_1 + x_2 + x_3 + x_4 + x_5 = 0$. □

Interactive Examples 14.8

Interactive solutions are on the web page http//www.math.ucsd.edu/˜ashenk/.†

1. What are the first-order partial derivatives of $F(x, y, z, w) = x^5 + y^4 + z^3 + w^2$?

2. Give an equation of the tangent plane to $w = xye^z$ at $x = 1, y = 2, z = 3$.

3. What is the gradient vector of $H(x, y, z) = \sin x + \sin y + \sin x + \sin z$ at (π, π, π, π)?

4. Use the result of Interactive Example 3 to find the derivative of $H(x, y, z) = \sin x + \sin y + \sin z + \sin w$ at (π, π, π, π) in the direction toward $(3\pi, 3\pi, 3\pi, 3\pi)$.

5. Find the maximum directional derivative of $K(x, y, z, w) = xyzw$ at $(1, 2, 3, 4)$.

6 Give an equation of the tangent plane to $xy + yz + zw + xw = 4$ at $(1, 1, 1, 1)$ in $xyzw$-space.

Exercises 14.8

AAnswer provided. **O**Outline of solution provided. **C**Graphing calculator or computer required.

CONCEPTS:

1. Suppose that the cost function $C(x_1, x_2, x_3, x_4, x_5)$ in Example 1 is defined for all positive values of its variables. What are the rates of change with respect to x_1, x_2, x_3, x_4 and x_5 of the cost function and what do they represent?

2. How can a function $w = f(x, y, z, u, v)$ be viewed as a function of the three variables (x, y, z)?

3. When does a function $f(x)$ of one variable become a function of the four variables (x, y, z, w)?

4. Show that the formula $w = f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c)$ for the tangent plane to the graph of a function can be obtained from the formula for the tangent plane to a level surface of $g(x, y, z, w) = f(x, y, z) - w$ at $\big(a, b, c, f(a, b, c)\big)$.

BASICS:

Find the derivatives in Exercises 5 through 11.

5.[O] g_{xyzw} for $g(x, y, z, w) = (\sin x)(\cos y)(\ln z)(e^{2w})$.

6.[A] $\dfrac{\partial}{\partial x}(x^2y^3z^4w^5)$

7. $\dfrac{\partial}{\partial w}(6x - 7y + 4z - 7w)$

8.[A] $\dfrac{\partial^2}{\partial z \partial w}(e^{5x+4y+3z+2w})$

9. $\dfrac{\partial^3}{\partial x \partial w^2}(x^2y^3z^4w^5)$

10.[A] P_{xyzw} for $P(x, y, z, w) = xyzw$

11. $\dfrac{\partial^3}{\partial w^3}(e^{x+2y+3z+4w})$

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†In the published text the interactive solutions of these examples will be on an accompanying CD disk which can be run by any computer browser without using an internet connection.

In Exercises 12 through 15 give equations of the tangent plnes.

12.$^{\mathsf{O}}$ The tangent plane to the graph of $w = \sin x \sin y \sin z$ at $x = 1, y = 2, z = 3$.

13.$^{\mathsf{A}}$ The tangent plane to the graph of $w = e^{xyz}$ at $x = 1, y = -1, z = -1$

14. The tangent plane to the graph of $w = e^x + e^y + e^z$ at $x = 1, y = 1, z = 1$

15. The tangent plane to the graph of $g = 5 + x^2 - y^2 + z^2 - w^2$ at $x = 0, y = 0, z = 0, w = 0$

Find the gradient vectors in Exercises 16 through 19.

16.$^{\mathsf{O}}$ The gradient vector of $f(x, y, z, w) = xyzw$ at $(1, 2, 3, 4)$

17.$^{\mathsf{A}}$ The gradient vector of $g(x, y, z, u, v) = 2x + 4y + 6z + 8u + 10v$.

18. $\nabla Q(0, 0, 0, 0)$ for $Q(x, y, z, w) = \sin x + \sin(2y) + \sin(3z) + \sin(4w)$?

19. $\nabla P(x, y, z, w)$ for $P(x, y, z, w) = x^{-4} + y^{-3} + z^{-2} + w^{-1}$?

Find the directional derivatives in Exercises 20 through 23.

20.$^{\mathsf{O}}$ The directional derivative of $f(x, y, z, w) = xyzw$ at $(1, 2, 3, 4)$ in the direction toward $(2, 3, 4, 5)$ (See Exercise 12.)

21.$^{\mathsf{A}}$ $D_{\mathbf{u}}g(x, y, z, u, v)$ for $g(x, y, z, u, v) = 2x+4y+6z+8u+10v$ and $\mathbf{u} = \left\langle \frac{3}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5} \right\rangle$ (See Exercise 13.)

22. The directional derivative of $H = \sqrt{xyzw}$ at $(2, 3, 2, 3)$ in the direction toward the origin

23. $D_{\mathbf{u}}h(0, 0, 0, 0, 0)$ for $H(x, y, z, u, v) = e^{x+y+z+u+v}$ and $\mathbf{u} = \dfrac{\langle 1, 1, 1, 1, 1 \rangle}{\sqrt{5}}$

In Exercises 24 through 27 give formulas for the tangent planes to the level surfaces.

24.$^{\mathsf{O}}$ The tangent plane to $\sqrt{x} + \sqrt{y} + \sqrt{z} + \sqrt{w} = 10$ at $(1, 2, 3, 4)$ in $\Re^4$

25.$^{\mathsf{A}}$ The tangent plane to $xyzw = 24$ at $(1, 2, 3, 4)$ in $xyzw$-space

26. The tangent plane to $\sin x + \sin y + \sin z + \sin w = 0$ at the origin in $\Re^4$

27. The tangent plane to $x + x^2 = 6$ at $(2, 3, 4, 5)$ in $xyzw$-space

(End of Section 14.8)