

The main tools of calculus—the derivative and the integral—are defined by using limits, and many of the basic results of calculus use the related concept of continuity. We discuss finite limits in Sections 1.1 and 1.2 and continuity in Section 1.3. Infinite limits are covered in Section 1.4, and formal definitions of finite and infinite limits are presented in Section 1.5.

Section 1.1

Finite limits

OVERVIEW: *This and the next section contain background material on the types of limits that will be used to define derivatives in Chapter 2.*

Topics:

- *A first look at instantaneous velocity*
- *One-sided finite limits*
- *A function without a limit*
- *Two-sided finite limits*
- *Limits of sums, products, powers, and quotients*
- *Limits of polynomials and rational functions*

A first look at instantaneous velocity

Suppose that a ball is dropped and there is no air resistance, so the only force on it is the downward force of gravity. The ball falls $h = 16t^2$ feet in t seconds (Figure 1). The graph of this height function is the half parabola in Figure 2. We want to determine how fast the ball is falling at $t = 1$ by finding its instantaneous velocity at that moment.

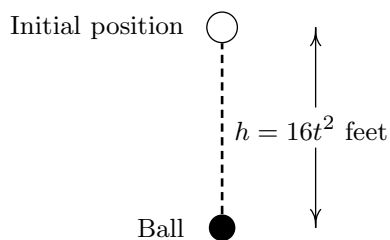


FIGURE 1

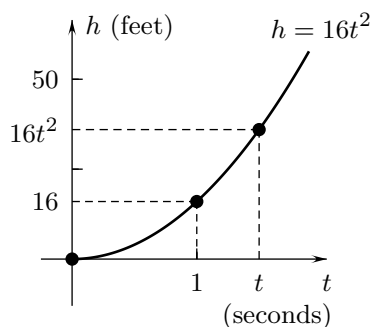


FIGURE 2

To do this, we first find a formula for the ball's AVERAGE VELOCITY in the interval from time 1 to a slightly later time t . Let us denote this average velocity by $A(t)$. At time 1 the ball has fallen 16 feet and at time t it has fallen $16t^2$ feet (Figure 2), so from time 1 to time t it travels $16t^2 - 16$ feet. This takes $t - 1$ seconds, so the ball's average velocity is

$$A(t) = \frac{\text{Distance traveled}}{\text{Time taken}} = \frac{16t^2 - 16 \text{ feet}}{t - 1 \text{ second}}. \quad (1)$$

We estimate the ball's instantaneous velocity at time 1 by calculating this average velocity for times t very close to 1, as in the following table.

TABLE 1. THE BALL'S AVERAGE VELOCITY FROM TIME 1 TO TIME t

t	1.1	1.01	1.001	1.0001	1.00001	1.000001
$A(t) = \frac{16t^2 - 16}{t - 1}$	33.6	32.16	32.016	32.0016	32.00016	32.000016

The average velocity (1) is not defined for $t = 1$, where the numerator and denominator are both zero. The calculations in Table 1 suggest that it gets closer to 32 as t gets closer to 1. In fact, it does approach 32 as t approaches 1, as is suggested by its graph in Figure 3. We say that 32 is the LIMIT of the average velocity and that the ball's INSTANTANEOUS VELOCITY at $t = 1$ is 32 feet per second. This approach to studying velocity will be the basis of the definition of the derivative in Chapter 2.

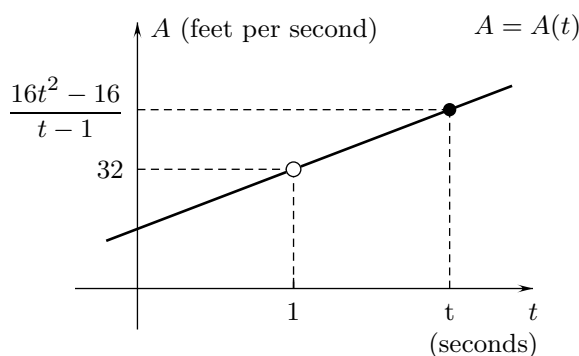


FIGURE 3

In the next section we will present a special technique that can be used to find limits of functions like the average velocity (1). Meanwhile, in this section, we will explain limit concepts with examples where the limits can be determined directly from the formulas for the functions.

One-sided finite limits

Figure 4 shows the graph of a function given by

$$F(x) = \begin{cases} 4 - x^2 & \text{for } x < 1 \\ 4 & \text{for } x = 1 \\ 2x & \text{for } x > 1. \end{cases} \quad (2)$$

This function is defined by three different formulas and its graph is in three pieces. The graph is formed by the portion of the parabola $y = 4 - x^2$ for $x < 1$, the point $(1, 4)$ at $x = 1$, and the portion of the line $y = 2x$ for $x > 1$. A dot has been placed at $(1, 4)$ to indicate that it is a point on the graph, and open circles have been placed at the right end of the parabola and the left end of the line to show that those points are not part of the graph.

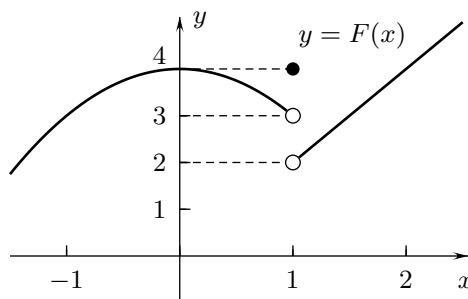


FIGURE 4

Example 1 Calculate the values of $F(x)$, defined by (2) at $x = 0.9, 0.99, 0.999$, and 0.999 , and at $x = 1.1, 1.01, 1.001$, and 1.0001 . Then describe the results.

SOLUTION The values are given on the table below. The values of F are calculated using the formula $F(x) = 4 - x^2$ in the first column where x is less than 1 and by $F(x) = 2x$ in the second column where x is greater than 1. The values of x in the first column are approaching 1 from the left and the corresponding values of $F(x)$ seem to approach 3. The values of x in the second column are approaching 1 from the right and the corresponding values of $F(x)$ seem to approach 2. \square

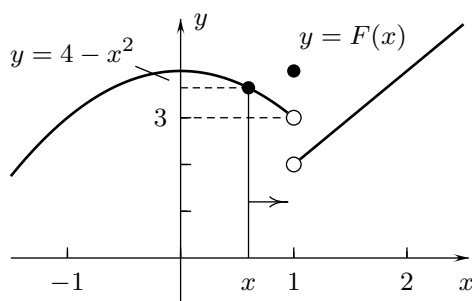
x	$F(x) = 4 - x^2$	x	$F(x) = 2x$
0.9	3.19	1.1	2.2
0.99	3.0199	1.01	2.02
0.999	3.001999	1.001	2.002
0.9999	3.00019999	1.0001	2.0002

As is suggested by Example 1, the value $F(x)$ approaches 3 as x approaches 1 from the left (see Figure 5) and approaches 2 as x approaches 1 from the right (see Figure 6). We say that the LIMIT OF $F(x)$ AS x APPROACHES 1 FROM THE LEFT is 3 and that the limit of $F(x)$ as x approaches 1 from the right is 2. We write

$$\lim_{x \rightarrow 1^-} F(x) = 3$$

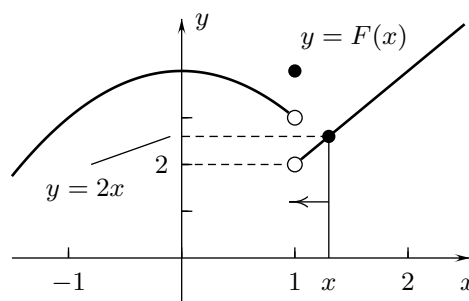
$$\lim_{x \rightarrow 1^+} F(x) = 2.$$

(Notice that the term “limit” in these statements does not mean “barrier” or “boundary” as is generally the case when this term is used in everyday English. The meaning of “limit” here is closer to that of “destination” in everyday speech.)



$$\lim_{x \rightarrow 1^-} F(x) = 3$$

FIGURE 5



$$\lim_{x \rightarrow 1^+} F(x) = 2$$

FIGURE 6

A minus sign is used as a superscript for limits from the left and a plus sign for limits from the right because the negative x -axis is on the left and the positive x -axis is on the right. We can also describe the limits of $F(x)$ as x approaches 1 from the left and right with the notation

$$F(x) \rightarrow 3 \text{ as } x \rightarrow 1^- \quad \text{and} \quad F(x) \rightarrow 2 \text{ as } x \rightarrow 1^+.$$

The value of the function at $x = 1$, namely $F(1) = 4$, is not considered in finding the limits of $F(x)$ as x tends to 1 from the left and from the right. This convention is followed so that the limits can be used to distinguish between the behavior of a function near a point and its value at the point and to deal with the especially important cases where functions are not defined at the limiting values of x . Here is the general definition of the limits:

Definition 1 (One-sided limits) (a) Suppose that $y = f(x)$ is defined on an open interval (b, a) to the left of a fixed point a . The number L is the **LIMIT OF $f(x)$ AS x APPROACHES a FROM THE LEFT** if $f(x)$ is arbitrarily close to the number L for all $x < a$ sufficiently close to a .[†] In this case, we write

$$L = \lim_{x \rightarrow a^-} f(x). \quad (3)$$

(b) Suppose that $y = f(x)$ is defined on an open interval (a, b) to the right of a . The number L is the **LIMIT OF $f(x)$ AS x APPROACHES a FROM THE RIGHT** if $f(x)$ is arbitrarily close to L for all $x > a$ sufficiently close to a . In this case, we write

$$L = \lim_{x \rightarrow a^+} f(x). \quad (4)$$

[†]The concepts of “arbitrarily close” and “sufficiently close” will be made precise in Section 1.5, where we discuss formal definitions of limits.

Example 2 Figure 7 shows the graph of

$$G(x) = \begin{cases} -\frac{1}{2}x^3 & \text{for } x \leq 2 \\ 8/x & \text{for } x > 2. \end{cases}$$

What are (a) $\lim_{x \rightarrow 2^-} G(x)$ and (b) $\lim_{x \rightarrow 2^+} G(x)$?

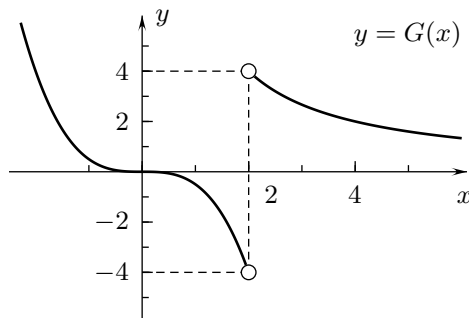


FIGURE 7

SOLUTION

(a) As x approaches 2 from the left in Figure 7, the point $(x, G(x))$ traces out the portion of the graph on the left and approaches the point with y -coordinate -4 at its left end. Therefore,

$$\lim_{x \rightarrow 2^-} G(x) = -4.$$

(b) Similarly, as x approaches 2 from the right, the point $(x, G(x))$ traces out the portion of the graph on the right and approaches the point with y -coordinate 4 at its right end. Consequently,

$$\lim_{x \rightarrow 2^+} G(x) = 4. \quad \square$$

A function without a limit

The sine function $y = \sin x$ has the value 1 at $x = \frac{1}{2}\pi$, at $x = \frac{5}{2}\pi$, at $x = \frac{9}{2}\pi$, and in general at $x = (2n + \frac{1}{2})\pi$ for all integers $n \geq 0$. On the other hand, it has the value -1 at $x = \frac{3}{2}\pi$, at $x = \frac{7}{2}\pi$, at $x = \frac{11}{2}\pi$, and in general at $x = (2n + \frac{3}{2})\pi$ for all integers $n \geq 0$. Consequently,

$$\sin\left(\frac{1}{x}\right) = \begin{cases} 1 \text{ at } x = \frac{1}{\frac{1}{2}\pi}, \frac{1}{\frac{5}{2}\pi}, \frac{1}{\frac{9}{2}\pi}, \dots \\ -1 \text{ at } x = \frac{1}{\frac{3}{2}\pi}, \frac{1}{\frac{7}{2}\pi}, \frac{1}{\frac{11}{2}\pi}, \dots \end{cases} \quad (5)$$

Since the values of x on the right of (5) are positive and tend to zero, the function $y = \sin(1/x)$ oscillates infinitely often between 1 and -1 as x approaches 0 from the right. This is shown by the portion of its graph in Figure 8. As a consequence, $\sin(1/x)$ does not approach any one number, and we say that it DOES NOT HAVE A LIMIT as x tends to 0 from the right.

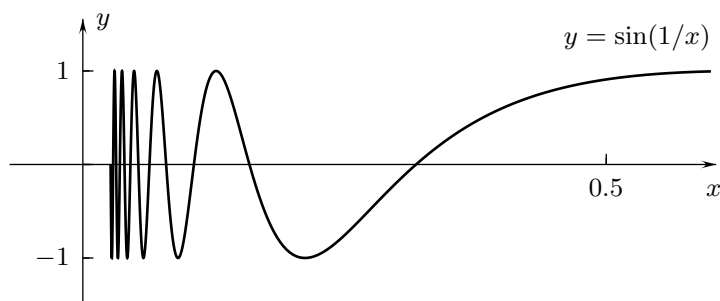


FIGURE 8

Two-sided finite limits

The LIMIT (or the TWO-SIDED LIMIT) of $f(x)$ as x tends to a is the number L if both one-sided-limits at a exist and equal L . In this case we write $\lim_{x \rightarrow a} f(x) = L$.

Definition 2 (Two-sided limits) If $y = f(x)$ is defined on open intervals (b, a) and (a, c) on both sides of a , then $\lim_{x \rightarrow a} f(x) = L$ if $\lim_{x \rightarrow a^-} f(x) = L$ and $\lim_{x \rightarrow a^+} f(x) = L$. This means that $f(x)$ is arbitrarily close to L for all $x \neq a$ sufficiently close to a .

If $\lim_{x \rightarrow a^+} f(x)$ and $\lim_{x \rightarrow a^-} f(x)$ are different or one or both do not exist, we say that $\lim_{x \rightarrow a} f(x) = L$ does not exist or is not defined.

Example 3 The graph of $J(x) = \begin{cases} x^2 & \text{for } x < 2 \\ 8 - 2x & \text{for } x > 2 \end{cases}$ is given in Figure 9. Find the limit of $J(x)$ as $x \rightarrow 2$.

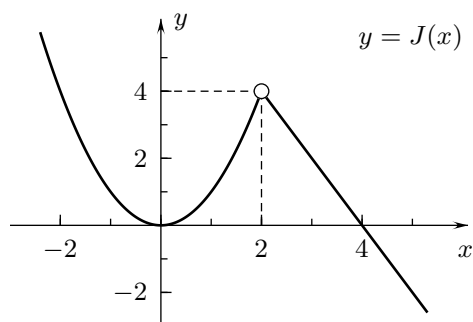


FIGURE 9

SOLUTION As x approaches 2 from the left in Figure 9, the point $(x, J(x))$ on the parabola approaches the point $(2, 4)$. Consequently,

$$\lim_{x \rightarrow 2^-} J(x) = 4.$$

As x approaches 2 from the right, the point $(x, J(x))$ on the line approaches the same point $(2, 4)$. Therefore,

$$\lim_{x \rightarrow 2^+} J(x) = 4.$$

Since both one-sided limits are 4, the two-sided limit is also 4: $\lim_{x \rightarrow 2} J(x) = 4$. \square

Limits of sums, products, powers, and quotients

Suppose that $f(x)$ tends to a number L and $g(x)$ tends to a number M as x tends to a . Then, as you can imagine, $f(x) + g(x)$ tends to $L + M$ as x tends to a (Figure 10). Similarly, $f(x)g(x)$ tends to LM , $[f(x)]^n$ tends to L^n for positive integers n , and $f(x)/g(x)$ tends to L/M if $M \neq 0$. These facts are stated in the following theorem, which is proved in advanced courses.

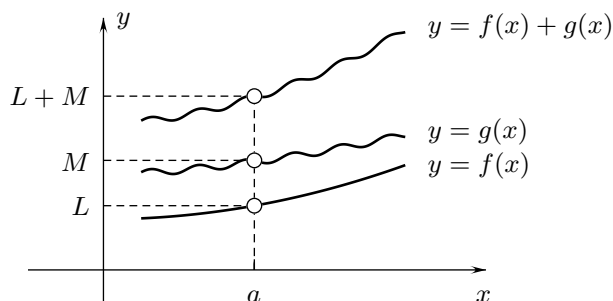


FIGURE 10

Theorem 1 (Limits of sums, products, powers, and quotients)

Suppose that $\lim_{x \rightarrow a} f(x) = L$ and $\lim_{x \rightarrow a} g(x) = M$ with numbers L and M . Then

$$\lim_{x \rightarrow a} [f(x) + g(x)] = L + M \quad (6)$$

$$\lim_{x \rightarrow a} [f(x)g(x)] = LM \quad (7)$$

and for positive integers n ,

$$\lim_{x \rightarrow a} [f(x)]^n = L^n. \quad (8)$$

Also, if $M \neq 0$, then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{L}{M}. \quad (9)$$

The two-sided limit in Theorem 1 can be replaced by limits from the left or from the right. Remember the results in the theorem verbally: "If two functions have finite limits, then the limit of their sum is the sum of their limits, the limit of their product is the product of their limits, the limit of a positive integer power of one of the functions is the power of its limit, and, if the limit of the denominator is not zero, the limit of their quotient is the quotient of their limits."

Example 4 What is $\lim_{x \rightarrow 5^+} \left[A(x)B(x) + \frac{C(x)}{D(x)} \right]$ if $\lim_{x \rightarrow 5^+} A(x) = 2$, $\lim_{x \rightarrow 5^+} B(x) = 5$, $\lim_{x \rightarrow 5^+} C(x) = 6$, and $\lim_{x \rightarrow 5^+} D(x) = 3$?

SOLUTION First, we use (7) with $x \rightarrow a$ replaced by $x \rightarrow 5^+$ and with $A(x)$ and $B(x)$ in place of $f(x)$ and $g(x)$ to write

$$\lim_{x \rightarrow 5^+} [A(x)B(x)] = \left[\lim_{x \rightarrow 5^+} A(x) \right] \left[\lim_{x \rightarrow 5^+} B(x) \right] = 2(5) = 10.$$

Next, we use (9) with $x \rightarrow a$ replaced by $x \rightarrow 5^+$ and with $C(x)$ and $D(x)$ in place of $f(x)$ and $g(x)$ to obtain

$$\lim_{x \rightarrow 5^+} \frac{C(x)}{D(x)} = \frac{\lim_{x \rightarrow 5^+} C(x)}{\lim_{x \rightarrow 5^+} D(x)} = \frac{6}{3} = 2.$$

Finally, by (6) with $x \rightarrow a$ replaced by $x \rightarrow 5^+$ and with $A(x)B(x)$ and $C(x)/D(x)$ in place of $f(x)$ and $g(x)$, we have

$$\lim_{x \rightarrow 5^+} \left[A(x)B(x) + \frac{C(x)}{D(x)} \right] = \lim_{x \rightarrow 5^+} [A(x)B(x)] + \lim_{x \rightarrow 5^+} \frac{C(x)}{D(x)} = 10 + 2 = 12. \quad \square$$

Limits of polynomials and rational functions

For any a , the limit as x tends to a of a constant function $y = C$ is C and the limit of the function $y = x$ is a :

$$\lim_{x \rightarrow a} C = C \quad \text{and} \quad \lim_{x \rightarrow a} x = a.$$

These facts with Theorem 1 lead to the next result, which enables us to find limits of polynomials and rational functions.

Theorem 2 (Limits of polynomials and rational functions) If $y = f(x)$ is a polynomial or is a rational function that is defined at a , then

$$\lim_{x \rightarrow a} f(x) = f(a)$$

and consequently

$$\lim_{x \rightarrow a^\pm} f(x) = f(a).$$

Example 5 Find $\lim_{x \rightarrow 10^+} (x^3 + 5x^2 + 6x)$.

SOLUTION Since $y = x^3 + 5x^2 + 6x$ is a polynomial, its limit as x tends to 10 from the right is its value at $x = 10$:[†]

$$\begin{aligned} \lim_{x \rightarrow 10^+} (x^3 + 5x^2 + 6x) &= [x^3 + 5x^2 + 6x]_{x=10} \\ &= 10^3 + 5(10^2) + 6(10) \\ &= 1000 + 500 + 60 = 1560. \quad \square \end{aligned}$$

[†]Recall that $[f(x)]_{x=a}$ denotes the value $f(a)$ of f at a .

Example 6 What is $\lim_{x \rightarrow 1} \frac{x+5}{x+2}$?

SOLUTION $y = \frac{x+5}{x+2}$ is a rational function that is defined for all x except at $x = -2$, where its denominator is zero. Consequently, it is defined at $x = 1$ and its limit as x tends to 1 is its value at $x = 1$:

$$\lim_{x \rightarrow 1} \frac{x+5}{x+2} = \left[\frac{x+5}{x+2} \right]_{x=1} = \frac{1+5}{1+2} = \frac{6}{3} = 2. \square$$

Theorem 2 can be used to find limits of functions given by two or more polynomials or rational functions in different intervals, as in the next examples.

Example 7 A function S is defined by

$$S(x) = \begin{cases} x^2 + x + 1 & \text{for } x < 1 \\ 5 & \text{for } x = 1 \\ 2 + 1/x^4 & \text{for } x > 1. \end{cases}$$

Does $\lim_{x \rightarrow 1} S(x)$ exist? If so, give its value.

SOLUTION Because $S(x)$ is equal to the polynomial $y = x^2 + x + 1$ for $x < 1$ and the limit of the polynomial as x tends to 1 from the left is its value at $x = 1$,

$$\lim_{x \rightarrow 1^-} S(x) = \lim_{x \rightarrow 1^-} (x^2 + x + 1) = \left[x^2 + x + 1 \right]_{x=1} = 1^2 + 1 + 1 = 3.$$

For $x > 1$, $S(x)$ is equal to the rational function $y = 2 + 1/x^4$. The rational function is defined at $x = 1$ because its denominator is not zero there. Consequently, its limit as x tends to 1 from the right is equal its value at that point, and

$$\lim_{x \rightarrow 1^+} S(x) = \lim_{x \rightarrow 1^+} \left(2 + \frac{1}{x^4} \right) = \left[2 + \frac{1}{x^4} \right]_{x=1} = 2 + \frac{1}{1^4} = 2 + 1 = 3.$$

Since the limits of $S(x)$ as x tends to 1 from the left and from the right are both 3, the two-sided limit is also 3: $\lim_{x \rightarrow 1} S(x) = 3$. \square

Example 8 Find $\lim_{x \rightarrow 1} T(x)$ where

$$T(x) = \begin{cases} x + 1 & \text{for } x \leq 1 \\ \frac{1}{x} & \text{for } x > 1. \end{cases}$$

SOLUTION Since $y = x + 1$ is a polynomial and $y = 1/x$ is a rational function defined at $x = 1$,

$$\begin{aligned} \lim_{x \rightarrow 1^-} T(x) &= \lim_{x \rightarrow 1^-} (x + 1) = \left[x + 1 \right]_{x=1} = x + 1 = 2 \\ \lim_{x \rightarrow 1^+} T(x) &= \lim_{x \rightarrow 1^+} \frac{1}{x} = \left[\frac{1}{x} \right]_{x=1} = 1. \end{aligned} \tag{10}$$

The two-sided limit $\lim_{x \rightarrow 1} T(x)$ is not defined because the one-sided limits (10) are different. This can be seen from the graph of the function in Figure 11: the two halves of the graph do not come together at $x = 1$. \square

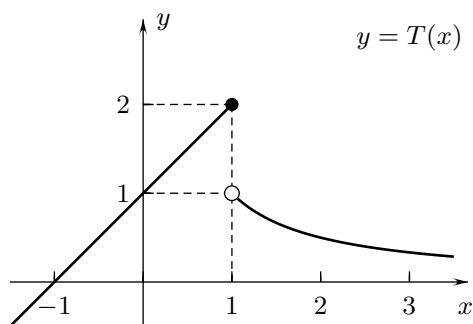


FIGURE 11

Interactive Examples 1.1

Interactive solutions are on the web page <http://www.math.ucsd.edu/~ashenk/>.[†]

1. Figure 12 shows the graph of a function f defined for $-1 \leq x \leq 3$. Find (a) $f(-1)$, (b) $\lim_{x \rightarrow -1^+} f(x)$, (c) $\lim_{x \rightarrow 1^-} f(x)$, (d) $f(1)$, and (e) $\lim_{x \rightarrow 1^+} f(x)$.

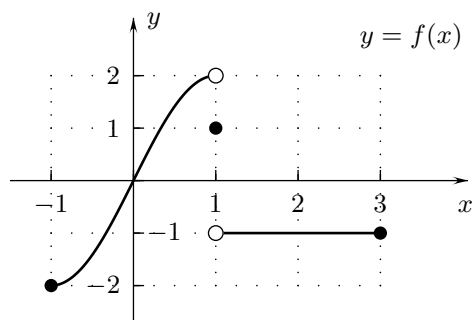


FIGURE 12

2. (a) Sketch the graph of $g(x) = \begin{cases} x & \text{for } x < 4 \\ 16/x & \text{for } x > 4. \end{cases}$ (b) What is $\lim_{x \rightarrow 4} g(x)$?
3. Without drawing the graph, find (a) $S(2)$, (b) $\lim_{x \rightarrow 2^-} S(x)$, (c) $\lim_{x \rightarrow 2^+} S(x)$, and (d) $\lim_{x \rightarrow 2} S(x)$ where

$$S(x) = \begin{cases} 8x^{-3} & \text{for } x < 2 \\ 3 & \text{for } x = 2 \\ \frac{1}{8}x^3 & \text{for } x > 2. \end{cases}$$

4. What is $\lim_{x \rightarrow 5} [x^2 g(x) + 2x]$ if $\lim_{x \rightarrow 5} g(x) = 10$?

[†]In the published text the interactive solutions of these examples will be on an accompanying CD disk which can be run by any computer browser without using an internet connection.

Exercises 1.1

^AAnswer provided. ^OOutline of solution provided. ^CGraphing calculator or computer required.

CONCEPTS:

1. Match the following statements to Figures 13 through 15 without using any statement or figure twice: (a) $\lim_{x \rightarrow x_0^+} f(x) = L$ (b) $\lim_{x \rightarrow x_0^-} f(x) = L$, (c) $\lim_{x \rightarrow x_0} f(x) = L$.

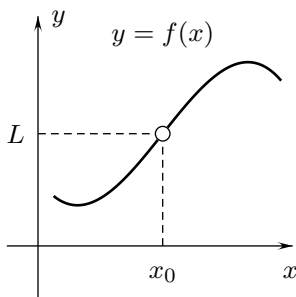


FIGURE 13

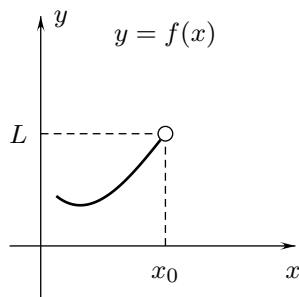


FIGURE 14

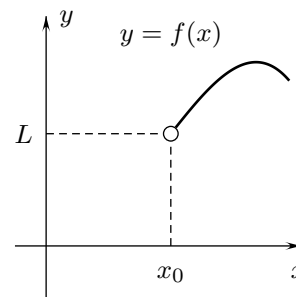


FIGURE 15

2. What are $\lim_{x \rightarrow 5^-} g(x)$ and $\lim_{x \rightarrow 5^+} g(x)$ if $\lim_{x \rightarrow 5} g(x) = 7$?
3. Sketch the graph of a function g such that $\lim_{x \rightarrow 1} g(x) = 2$ and $g(x) = 3$.
4. Draw the graph of a function h such that $\lim_{x \rightarrow 0^-} h(x) = 2$, $h(0) = 4$, and $\lim_{x \rightarrow 0^+} h(x) = 6$.
5. What is $\lim_{x \rightarrow 5} K(x)$ if $\lim_{x \rightarrow 5} L(x) = 10$ and $K(x) = L(x)$ for $x \neq 5$?
6. Suppose that $y = A(x)$ is defined for all $x \neq 2$, that $\lim_{x \rightarrow 2^+} A(x) = 3$, and that $\lim_{x \rightarrow 2^-} A(x) = 10$. Pick a constant k such that $\lim_{x \rightarrow 2} B(x)$ exists where $B(x) = A(x)$ for $x < 2$ and $B(x) = A(x) + k$ for $x > 2$.

BASICS:

- 7.^O Figure 16 shows the graph of a function f defined for $-4 \leq x \leq 4$. Find (a) $\lim_{x \rightarrow -4^+} f(x)$, (b) $\lim_{x \rightarrow 0^-} f(x)$, (c) $\lim_{x \rightarrow 0^+} f(x)$, (d) $\lim_{x \rightarrow 0} f(x)$, (e) $\lim_{x \rightarrow 2} f(x)$, and (f) $\lim_{x \rightarrow 4^-} f(x)$.

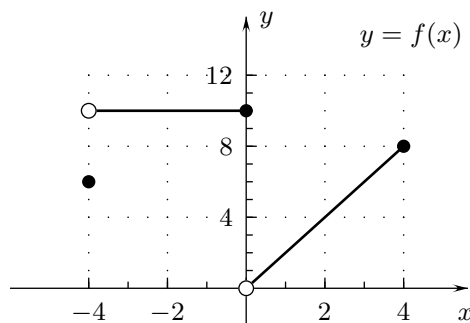


FIGURE 16

- 8.^O Find $\lim_{x \rightarrow 2} (1 + 2x^2 + x^3)$.
- 9.^O What is $\lim_{x \rightarrow 0^+} \left(\frac{2x + 4}{x + 2} \right)$?

- 10.⁰** (a) Sketch the graph of the function P defined below. Then find (b) $P(-1)$, (c) $\lim_{x \rightarrow -1^-} P(x)$, (d) $\lim_{x \rightarrow -1^+} P(x)$, (e) $P(0)$, and (f) $\lim_{x \rightarrow 0} P(x)$.

$$P(x) = \begin{cases} -3/x & \text{for } x < -1 \\ 1 & \text{for } -1 \leq x \leq 1 \\ 3/x & \text{for } x > 1. \end{cases}$$

Find the limits in Exercises 11 through 21. Justify your answers.

- 11.⁰** $\lim_{x \rightarrow 3^+} (x + x^2 + x^3)$ **13.** $\lim_{x \rightarrow 1^+} \left(\frac{x^2 + 1}{x^2 + 4} \right)$
- 12.^A** $\lim_{x \rightarrow 4} \left(x^2 + \frac{6}{x+2} \right)$ **14.** $\lim_{x \rightarrow 0^-} \frac{10 + x^2 + x^3}{2 - x - x^5}$
- 15.⁰** $\lim_{x \rightarrow 7} \frac{F(x) + G(x)}{F(x) - G(x)}$ where $\lim_{x \rightarrow 7} F(x) = 20$ and $\lim_{x \rightarrow 7} G(x) = 10$
- 16.^A** $\lim_{x \rightarrow 0} ([g(x)]^3 + [g(x)]^2 + g(x))$ where $\lim_{x \rightarrow 0} g(x) = 10$
- 17.** The limit of $P(x)Q(x)$ as $x \rightarrow 5^-$ where $\lim_{x \rightarrow 5^-} P(x) = 3$ and $\lim_{x \rightarrow 5^-} Q(x) = 7$
- 18.** $\lim_{x \rightarrow 100^+} \frac{1 + r(x)}{2 + r(x)}$ where $\lim_{x \rightarrow 100^+} r(x) = -1$
- 19.⁰** $\lim_{x \rightarrow 4} [xh(x) + x^2]$ where $\lim_{x \rightarrow 4} h(x) = 10$
- 20.^A** $\lim_{x \rightarrow 6} \left(\frac{x + f(x)}{8 - x} \right)^3$ where $\lim_{x \rightarrow 6} f(x) = 4$
- 21.** $\lim_{x \rightarrow 0} \frac{U(x) + V(x)}{x - V(x)}$ where $\lim_{x \rightarrow 0} U(x) = 10$ and $\lim_{x \rightarrow 0} V(x) = -10$
- 22.⁰** (a) Sketch the graph of

$$w(x) = \begin{cases} x^2 & \text{for } 0 < x < 2 \\ 2 & \text{for } x > 2. \end{cases}$$

Find the limits of $w(x)$ (b) as $x \rightarrow 0^+$, (c) as $x \rightarrow 1$, (d) as $x \rightarrow 2^-$, (e) as $x \rightarrow 2^+$, and (f) as $x \rightarrow 3$. (g) Find all solutions x of $w(x) = 3$.

- 23.^A** (a) Sketch the graph of

$$T(x) = \begin{cases} x & \text{for } x < 1 \\ x - 2 & \text{for } x \geq 1. \end{cases}$$

(b) Find $T(1)$, $\lim_{x \rightarrow 1^-} T(x)$, and $\lim_{x \rightarrow 1^+} T(x)$. (c) What are the greatest and least values of T for $0 \leq x \leq 4$ and at what values of x do they occur?

- 24.** (a) Sketch the graph of

$$K(x) = \begin{cases} 2 + x & \text{for } x \leq -1 \\ -x^2 & \text{for } -1 < x < 1 \\ 2 - x & \text{for } x \geq 1. \end{cases}$$

Find the limits of $K(x)$ (b) as $x \rightarrow -1^-$, (c) as $x \rightarrow -1^+$, and (d) as $x \rightarrow -1$. (e) Solve $K(x) = 0$ for x . (f) Give the four open intervals in which $K(x)$ negative.

- 25.⁰** Find the limits (a) as $x \rightarrow 1^-$, (b) as $x \rightarrow 1^+$, and (c) as $x \rightarrow 1$ of

$$h(x) = \begin{cases} x^4 - 2x & \text{for } x < 1 \\ 5x^3 - 4x^2 & \text{for } x > 1. \end{cases}$$

- 26.^A** What is $\lim_{x \rightarrow 5} M(x)$ if

$$M(x) = \begin{cases} x^2 + x & \text{for } x < 5 \\ 0 & \text{for } x = 5 \\ 6x & \text{for } x > 5? \end{cases}$$

- 27.** Find (a) $\lim_{x \rightarrow 10} Z(x)$ and (b) $\lim_{x \rightarrow 20} Z(x)$, where

$$Z(x) = \begin{cases} x^2 + 900 & \text{for } x < 10 \\ 500 & \text{for } x = 10 \\ x^3 & \text{for } x > 10. \end{cases}$$

EXPLORATION:

- 28.** Predict $\lim_{x \rightarrow 1} \frac{x^\pi - 1}{x - 1}$ by calculating the approximate decimal values of the function at $x = 1.1, 1.001, 1.00001$, and 1.0000001 .
- 29.** Predict a formula for $\lim_{x \rightarrow 0} \frac{\sin(kx)}{x}$ by calculating values for x near 0 with several choices of the constant k .
- 30.^A** For what values of the constants a and b does $\lim_{x \rightarrow 2} Z(x)$ exist if

$$Z(x) = \begin{cases} ax^2 & \text{for } x < 2 \\ bx^3 & \text{for } x > 2? \end{cases}$$

- 31.** At what values of x is $y = \frac{x^2 - 1}{|x^2 - 1|}$ not defined, and what are its limits from the left and from the right at those points?

(End of Section 1.1)