

## Section 1.3

### Continuity

(3/19/08)

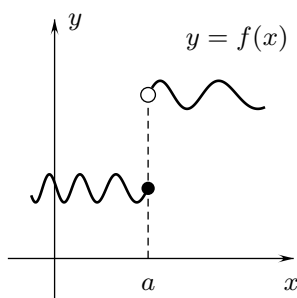
OVERVIEW: In Section 1.1 we studied one- and two-sided finite limits. In this section we discuss one- and two-sided continuity and continuity on intervals. These concepts will be needed in later chapters for discussions of optimization problems, the definition of definite integrals, and other basic topics. Then we present the Intermediate Value Theorem, which is used in studying equations involving functions that are continuous on finite, closed intervals.

Topics:

- **Continuity at a point**
- **Continuity of polynomials and rational functions**
- **Continuity on intervals**
- **Continuity of functions given by single formulas**
- **The Intermediate Value Theorem**

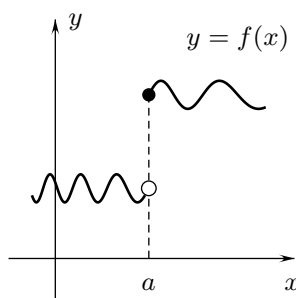
#### Continuity at a point

The function in Figure 1 is CONTINUOUS FROM THE LEFT at the point  $a$ , the function in Figure 2 is CONTINUOUS FROM THE RIGHT at  $a$ , and the function of Figure 3 CONTINUOUS at  $a$ , according to the following definition.



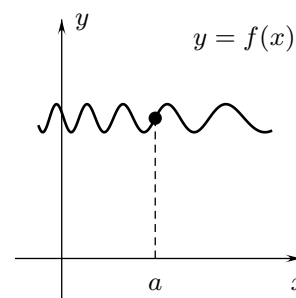
$f$  is continuous  
from the left at  $a$ .

FIGURE 1



$f$  is continuous  
from the right at  $a$ .

FIGURE 2



$f$  is continuous  
at  $a$ .

FIGURE 3

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#### Definition 1 (One-sided and two-sided continuity at a point)

(a)  $y = f(x)$  is CONTINUOUS FROM THE LEFT at  $a$  if it is defined in an interval  $(b, a]$  to the left of  $a$ , if  $\lim_{x \rightarrow a^-} f(x)$  exists, and if

$$\lim_{x \rightarrow a^-} f(x) = f(a).$$

(b)  $y = f(x)$  is CONTINUOUS FROM THE RIGHT at  $a$  if it is defined in an interval  $[a, b)$  to the right of  $a$ , if  $\lim_{x \rightarrow a^+} f(x)$  exists, and if

$$\lim_{x \rightarrow a^+} f(x) = f(a).$$

(c)  $y = f(x)$  is CONTINUOUS at  $a$  if it is defined in an open interval containing  $a$  and is continuous from the left and from the right at  $a$ . This means that  $\lim_{x \rightarrow a} f(x)$  exists and

$$\lim_{x \rightarrow a} f(x) = f(a).$$

In each case of Definition 1, three things must happen: the function must be defined at  $a$ , the limit must exist, and the limit must equal the value of the function.

In the case of continuity from the left in Figure 1, the value of the function at  $a$  is given by the dot, and the point on the graph for  $x < a$  approaches the dot as  $x$  approaches  $a$  from the left.

In the case of continuity from the right in Figure 2, the value of the function at  $a$  is given by the dot, and the point on the graph for  $x > a$  approaches the dot as  $x$  approaches  $a$  from the right.

In the case of two-sided continuity in Figure 3, the value of the function at  $a$  is given by the dot, and the point on the graph for  $x \neq a$  approaches the dot as  $x$  approaches  $a$  from either side.

Notice that a function is continuous at a point if it is continuous from the left and from the right at that point.

**Example 1** Figure 4 shows the graph of a function  $K$ , defined by

$$K(x) = \begin{cases} x + 4 & \text{for } -2 \leq x < 1 \\ x + 1 & \text{for } 1 \leq x \leq 4. \end{cases}$$

Use the graph to determine **(a)** the values  $a$  with  $-2 \leq a \leq 4$  where  $y = K(x)$  is continuous and **(b)** whether the function is continuous from the right or from the left at the points in  $[-2, 4]$  where it is not continuous.

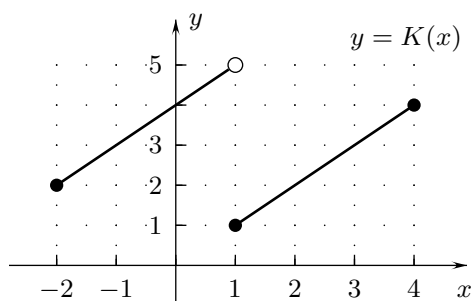


FIGURE 4

SOLUTION

**(a)** The function  $K$  is continuous at all  $a$  with  $-2 < a < 1$  and  $1 < a < 4$ , because  $\lim_{x \rightarrow a} K(x) = K(a)$  for all such  $a$ .

**(b)** The function is not continuous at  $-2$  because it is not defined for  $x < -2$ . It is, however, continuous from the right at  $-2$  since  $\lim_{x \rightarrow -2^+} K(x) = 2$  and  $K(-2) = 2$  are equal.

$K$  is not continuous at  $1$  since  $\lim_{x \rightarrow 1^-} K(x) = 5$  is not equal to  $K(1) = 1$ , but it is continuous from the right at  $1$  since  $\lim_{x \rightarrow 1^+} K(x) = 1$  and  $K(1) = 1$  are equal.

The function is not continuous at  $4$  because it is not defined for  $x > 4$ , but it is continuous from the left at  $4$  since  $\lim_{x \rightarrow 4^-} K(x) = 4$  and  $K(4) = 4$  are equal.  $\square$

### Continuity of polynomials and rational functions

According to Theorem 2 of Section 1.1, if  $f$  is a polynomial or is a rational function that is defined at  $a$ , then

$$\lim_{x \rightarrow a} f(x) = f(a).$$

This means that  $f$  is continuous at  $a$ , and we have the following result.

**Theorem 1 (Continuity of polynomials and rational functions)** *Polynomials are continuous at all values of  $x$ . Rational functions are continuous at all values of  $x$  where they are defined.*

**Example 2** At what values of  $x$  is  $y = \frac{x}{x^2 - 4}$  continuous?

**SOLUTION** Since the rational function  $y = \frac{x}{x^2 - 4}$  is defined except at  $x = 2$  and  $x = -2$ , where its denominator  $x^2 - 4$  is zero, it is continuous at all  $x \neq \pm 2$ . This property is illustrated by its graph in Figure 5.  $\square$

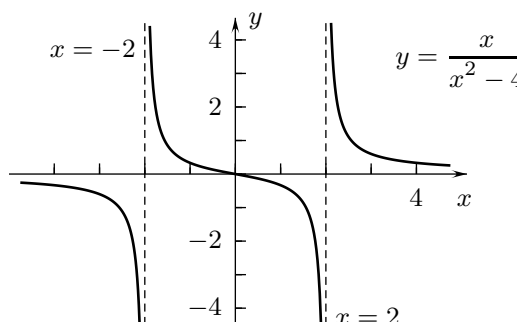


FIGURE 5

### Continuity on intervals

Theorem 1 is all we need to study continuity of polynomials and rational functions because their domains consist of open intervals (intervals that do not include their endpoints). We need, however, a modified definition of continuity to deal with functions whose domains include intervals that are not open.

Suppose, for example, we want to study the width  $w$  of a square as a function of its area  $A \geq 0$ . Since  $A = w^2$ , the width is given by the function  $w = \sqrt{A}$ , whose graph is shown in Figure 6. Its domain is the closed interval  $[0, \infty)$ . This function does not have a two-sided limit at  $A = 0$  because it is not defined for  $A < 0$ . Consequently, it is not continuous at  $A = 0$ . Nevertheless, it is continuous at all positive values of  $A$  and is continuous from the right at  $A = 0$ . We say that it is CONTINUOUS ON THE INTERVAL  $[0, \infty)$ , according to the following definition.

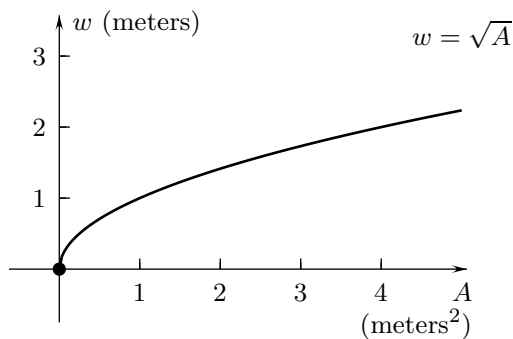
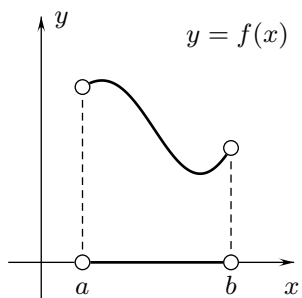


FIGURE 6

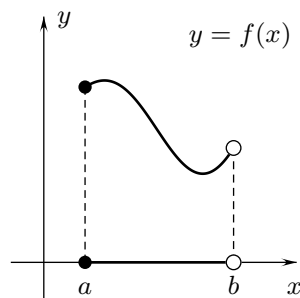
**Definition 2 (Continuity on an interval)** A function is continuous on an interval provided (i) it is continuous at each point in the interior<sup>†</sup> of the interval, (ii) it is continuous from the right at  $x = a$  if the interval includes a left endpoint  $a$ , and (iii) it is continuous from the left at  $x = b$  if the interval includes a right endpoint  $b$ .

Figures 7 through 10 illustrate this definition. The function of Figure 7 is continuous on the open interval  $(a, b)$  because it is continuous at each point in the interval and the interval does not include any endpoints. The function of Figure 8 is continuous on  $[a, b)$  because it is continuous at each point in the interior  $(a, b)$  of the interval and is continuous from the right at  $x = a$ . The function of Figure 9 is continuous on  $(a, b]$  because it is continuous at each point in the interior and is continuous from the left at  $x = b$ . The function of Figure 10 is continuous on  $[a, b]$  because it is continuous at each point in the interior and is continuous from the right at  $x = a$  and from the left at  $x = b$ .



$f$  is continuous  
on  $(a, b)$

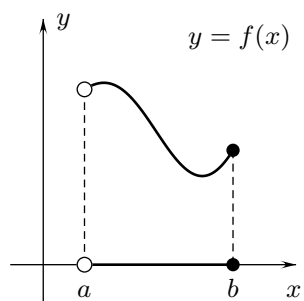
FIGURE 7



$f$  is continuous  
on  $[a, b)$

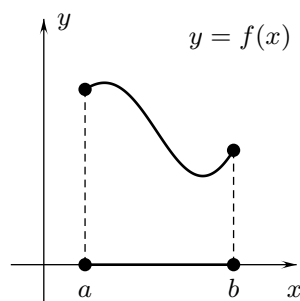
FIGURE 8

<sup>†</sup>Recall that the “interior” of an interval is obtained by removing its endpoints if it includes any.



$f$  is continuous  
on  $(a, b)$

FIGURE 9



$f$  is continuous  
on  $[a, b]$

FIGURE 10

**Example 3** Figure 11 shows the graph of a function  $M$  defined by

$$M(x) = \begin{cases} x + 3 & \text{for } -3 \leq x \leq -1 \\ x^3 + 1 & \text{for } -1 < x < 1 \\ 3 & \text{for } x = 1 \\ 2 & \text{for } 1 < x \leq 3. \end{cases}$$

Use the graph **(a)** to explain why  $M$  is not continuous on its domain  $[-3, 3]$  and **(b)** to find the largest intervals<sup>†</sup> on which it is continuous.

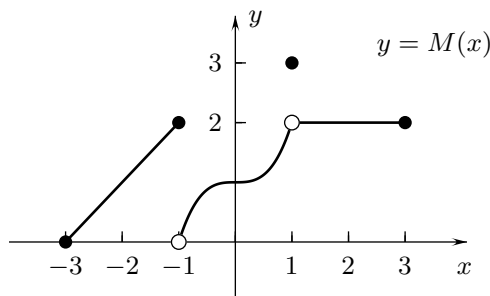


FIGURE 11

SOLUTION

**(a)**  $M$  is not continuous on  $[-3, 3]$  because it is not continuous at  $x = -1$  or at  $x = 1$  in the interior  $(-3, 3)$  of the interval.

**(b)**  $M$  is continuous on  $[-3, -1]$  because it is continuous at all  $x$  with  $-3 < x < -1$ , is continuous from the right at  $x = -3$ , and is continuous from the left at  $x = -1$ . It is continuous on the open interval  $(-1, 1)$  because it is continuous at each  $x$  with  $-1 < x < 1$ . It is continuous on  $(1, 3]$  because it is continuous at each  $x$  with  $1 < x < 3$  and is continuous from the left at  $x = 3$ . It is not continuous on any larger intervals because extending any of these intervals to the right or left would include points out of the domain of the function or put points of discontinuity in the interior of the interval.  $\square$

<sup>†</sup>When we say that one (possibly infinite) interval  $I_1$  is “larger” than another interval  $I_2$ , we mean that  $I_1$  contains  $I_2$  and is not equal to  $I_2$ .

### Continuity of functions given by single formulas

The function  $M$  of Example 3, which is not continuous on its domain  $[-3, 3]$ , is given by different formulas on different intervals. Because of the next theorem, which is established in advanced courses, almost any function in this text that is given by one formula on all its domain is continuous on the intervals where it is defined.

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**Theorem 2 (Continuity on intervals)** (a) The basic functions—power functions, the absolute value function, exponential functions, logarithms, trigonometric functions, inverse trigonometric function, and hyperbolic functions—are continuous on the intervals where they are defined.

(b) Any function that is given by a single formula formed from the functions of part (a) by taking linear combinations, products, quotients, or compositions is continuous on the intervals where it is defined.

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**Example 4** What are the largest intervals on which (a)  $y = \frac{3 \sin(\pi x)}{x^2 + 1}$  of Figure 12, (b)  $y = \sqrt{x^2 - 1}$  of Figure 13, and (c)  $y = \frac{x - 1}{|x - 1|}$  of Figure 14 are continuous?

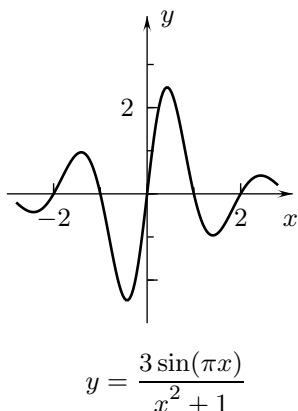


FIGURE 12

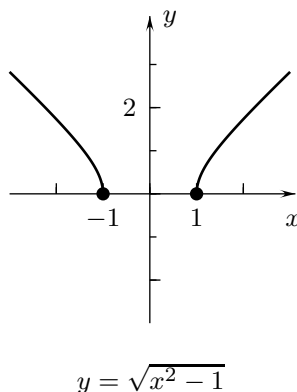


FIGURE 13

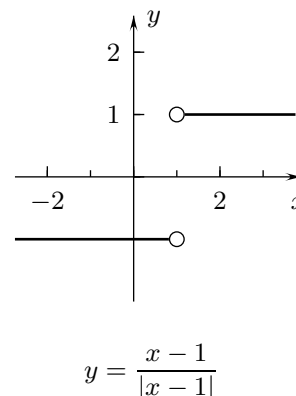


FIGURE 14

**SOLUTION**

(a)  $y = 3 \sin(\pi x)/(x^2 + 1)$  is given by one formula constructed from a power function, constant functions, and the sine function. It is defined for all  $x$ . By Theorem 2, it is continuous on  $(-\infty, \infty)$ .

(b)  $y = \sqrt{x^2 - 1}$  is given by a single formula formed from the power function  $y = x^2$ , the constant function  $y = 1$ , and the square-root function. Because of the square root, the function is defined where  $x^2 - 1 \geq 0$ , which is where  $x^2 \geq 1$ . Its domain is the union of the two intervals  $(-\infty, -1]$  and  $[1, \infty)$  and, by Theorem 2, it is continuous on these intervals.

(c)  $y = (x - 1)/|x - 1|$  is given by one formula constructed from the functions  $y = 1$ ,  $y = x$ , and  $y = |x|$ . Because it is defined except at  $x = 1$  where its denominator is zero, its domain consists of the open intervals  $(-\infty, 1)$  and  $(1, \infty)$  and, by Theorem 2, it is continuous on these intervals.  $\square$

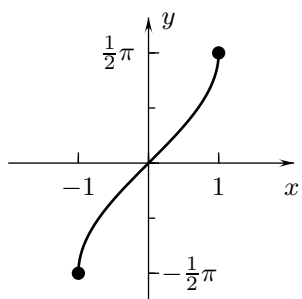
Theorem 2 can often be used to find limits, as in the next two examples.

**Example 5** What is the limit of  $y = 2x \sin^{-1} x$  as  $x$  tends to 1 from the left?

**SOLUTION**

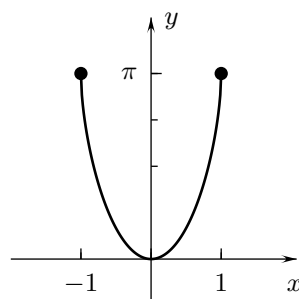
The inverse-sine function  $y = \sin^{-1} x$  is defined on  $[-1, 1]$  (Figure 15), and  $y = 2x$  is defined for all  $x$ . Therefore,  $y = 2x \sin^{-1} x$  (Figure 16) is also defined on  $[-1, 1]$ . Since it is given by a single formula formed from the basic functions, it is continuous on  $[-1, 1]$  by Theorem 2. In particular,  $y = 2x \sin^{-1} x$  is continuous from the left at  $x = 1$ , so its limit as  $x \rightarrow 1^-$  equals its value at  $x = 1$ , and

$$\lim_{x \rightarrow 1^-} (2x \sin^{-1} x) = \left[ 2x \sin^{-1} x \right]_{x=1} = 2(1) \sin^{-1}(1) = 2\left(\frac{1}{2}\pi\right) = \pi. \quad \square$$



$$y = \sin^{-1} x$$

FIGURE 15



$$y = 2x \sin^{-1} x$$

FIGURE 16

**Example 6** Figure 17 shows the graph of a function  $N$ , defined by

$$N(x) = \begin{cases} 2 \sin x & \text{for } x < 2 \\ 6 - e^{x/3} & \text{for } x > 2. \end{cases}$$

What are (a)  $\lim_{x \rightarrow 2^-} N(x)$  and (b)  $\lim_{x \rightarrow 2^+} N(x)$ ?

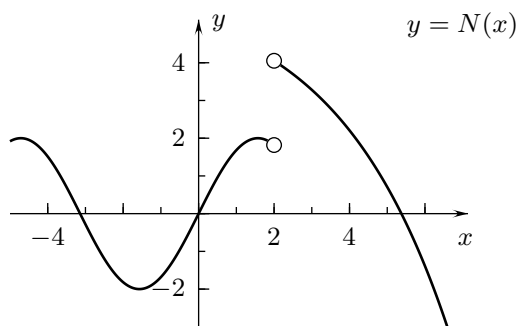


FIGURE 17

**SOLUTION**

(a) The functions  $y = 2 \sin x$  and  $y = 6 - e^{x/3}$  are continuous on  $(-\infty, \infty)$  because they are given by single formulas constructed from the basic functions and are defined for all  $x$ . Since  $N(x) = 2 \sin x$  for  $x < 2$ ,

$$\lim_{x \rightarrow 2^-} N(x) = \lim_{x \rightarrow 2^-} (2 \sin x) = \left[ 2 \sin x \right]_{x=2} = 2 \sin(2).$$

(b) Similarly, since  $N(x) = 6 - e^{x/3}$  for  $x > 2$ ,

$$\lim_{x \rightarrow 2^+} N(x) = \lim_{x \rightarrow 2^+} (6 - e^{x/3}) = \left[ 6 - e^{x/3} \right]_{x=2} = 6 - e^{2/3}. \quad \square$$

### The Intermediate Value Theorem

We close this section with a discussion of a theorem which is used to show, without solving the equations, that certain equations of the form  $f(x) = k$  with constant  $k$  have solutions  $x$ .

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**Theorem 3 (The Intermediate Value Theorem)** *If  $y = f(x)$  is continuous on a finite closed interval  $[a, b]$ , then the equation  $f(x) = k$  has at least one solution  $x$  with  $a \leq x \leq b$  for every number  $k$  between  $f(a)$  and  $f(b)$ .*

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This theorem is illustrated in Figure 18. In this case, the equation  $f(x) = k$  has a solution  $x$  in the interval  $[a, b]$  because the function  $y = f(x)$  is continuous on  $[a, b]$ ; the interval  $[a, b]$  is closed and finite; and  $k$  is between the values  $f(a)$  and  $f(b)$  of  $f$ .

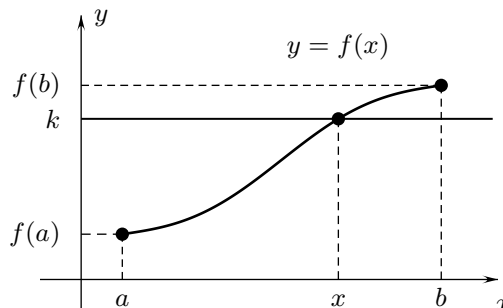


FIGURE 18

Theorem 3 can be justified informally as follows: because the function  $f$  is continuous on  $[a, b]$ , its graph for  $a \leq x \leq b$  cannot jump over the horizontal line  $y = k$  as  $x$  increases from  $a$  to  $b$  for any  $k$  between  $f(a)$  and  $f(b)$ . A formal proof of the theorem requires basic axioms of mathematics and is given in advanced courses.

This theorem, like all theorems, has a HYPOTHESIS and a CONCLUSION. The hypothesis is that  $f$  is continuous on  $[a, b]$ , and the conclusion is that  $f(x) = k$  has at least one solution in  $[a, b]$  for every  $k$  between  $f(a)$  and  $f(b)$ . In each instance where we might use the theorem, there are three possibilities:

- (i) The hypothesis is satisfied, in which case the conclusion must hold.
- (ii) The hypothesis is not satisfied and the conclusion does not hold.
- (iii) The hypothesis is not satisfied but the conclusion, nevertheless, holds.

The three possibilities are illustrated by the functions  $y = P(x)$ ,  $y = Q(x)$ , and  $y = R(x)$  of Figures 19, 20, and 21 in the interval  $[-1, 2]$ . (There is no fourth possibility since the conclusion cannot be false if the hypothesis is true.)

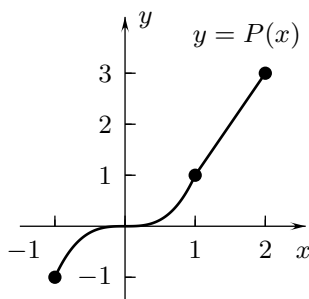


FIGURE 19

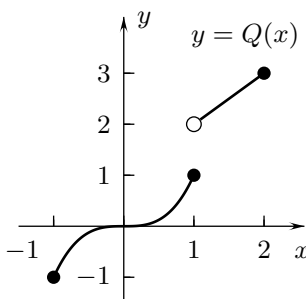


FIGURE 20

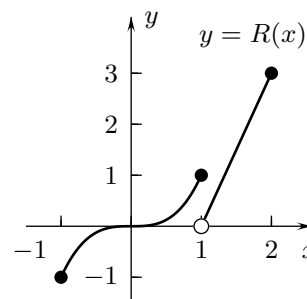


FIGURE 21

The function  $P$  of Figure 19 illustrates case (i). It satisfies the hypothesis of Theorem 3 because it is continuous on  $[-1, 2]$  and  $[-1, 2]$  is a finite, closed interval. It satisfies the conclusion because every horizontal line  $y = k$  with  $k$  between  $P(-1) = -1$  and  $P(2) = 3$  intersects the graph, so that the equation  $P(x) = k$  has a solution in  $[-1, 2]$  for all such  $k$ .

The function  $Q$  of Figure 20 illustrates case (ii). The hypothesis of the theorem is not satisfied by  $Q$  in  $[-1, 2]$  because  $Q$  is discontinuous at  $x = 1$ . The conclusion does not hold because the equation  $Q(x) = k$  does not have any solutions for  $1 < k \leq 2$ . The discontinuous  $Q$  jumps over these values of  $k$  as  $x$  increases from  $-1$  to  $2$ .

Case (iii) is illustrated by  $R$  of Figure 21. The hypothesis of the theorem is not satisfied by  $R$  in  $[-1, 2]$  because  $R$  is discontinuous at  $x = 1$ . The conclusion, however, does hold because all horizontal lines  $y = k$  with  $-1 \leq k \leq 3$  intersect the graph. In fact,  $R(x) = k$  has one solution  $x$  for each  $k$  with  $-1 \leq k \leq 0$  or  $1 < k \leq 3$  and two solutions for  $0 < k \leq 1$ , as can be seen from Figure 21.

The next example shows how the Intermediate Value Theorem can be used.

**Example 7** Use the Intermediate Value Theorem to show that the equation

$$3x + 7 \sin x = 10 \quad (1)$$

has at least one solution.

**SOLUTION** We set  $f(x) = 3x + 7 \sin x$ . After some experimentation, we notice that  $f$  has the value  $f(0) = 3(0) + 7 \sin(0) = 0$  at  $x = 0$  and the value  $f(6) = 18 + 7 \sin(6) \doteq 16.044$  at  $x = 6$ . Since the polynomial  $f$  is continuous on the interval  $[0, 6]$  and  $0 < 10 < 18 + 7 \sin(6)$ , we conclude from Theorem 3 that (1) has at least one solution  $x$  in  $[0, 6]$ . (In fact, the equation has three solutions in the interval, as can be seen from the graph of the function in Figure 22.)  $\square$

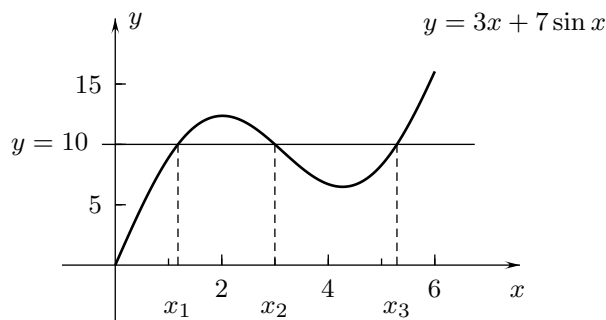


FIGURE 22

A calculator or computer could be used to find the approximate values  $x_1 \approx 1.17784222$ ,  $x_2 \approx 2.99690461$ , and  $x_3 \approx 5.28917686$  of the solutions of  $3x + 7 \sin x = 10$  in Figure 22.

**Example 8** Use the Intermediate Value Theorem to show that the equation  $x + 2x^2 + 3x^3 + 4x^4 = 10,000$  has at least one solution.

**SOLUTION** We set  $f(x) = x + 2x^2 + 3x^3 + 4x^4$ . After some experimentation, we find that  $f$  has the value 0 at  $x = 0$ , and the value 18,056 at  $x = 8$ . Since  $f$  is a polynomial it is continuous on  $[0, 8]$ . Moreover, 10,000 is between  $f(0)$  and  $f(8)$ . Therefore, by Theorem 3, the equation  $f(x) = 10,000$  has at least one solution  $x$  in  $[0, 8]$ .  $\square$

### Interactive Examples 1.3

Interactive solutions are on the web page <http://www.math.ucsd.edu/~ashenk/>.<sup>†</sup>

- At what values of  $x$  is  $y = \frac{x}{x+1} + \frac{6}{x^2}$  continuous?
- (a) Sketch the graph of the function

$$T(x) = \begin{cases} x+1 & \text{for } x < 0 \\ 1-x & \text{for } 0 \leq x \leq 1 \\ x & \text{for } x > 1. \end{cases}$$

- What are the largest intervals on which  $f(x) = e^{\sqrt{x}}$  is continuous? Use the graph to find (b) the points where  $T$  is continuous and (c) the largest intervals on which it is continuous.
- Give the largest intervals on which  $y = \frac{\ln x}{x^2 - 4}$  is continuous.
- What is  $\lim_{x \rightarrow 0^+} e^{\sqrt{x}}$ ?

### Exercises 1.3

<sup>A</sup>Answer provided. <sup>O</sup>Outline of solution provided. <sup>C</sup>Graphing calculator or computer required.

#### CONCEPTS:

- What is  $\lim_{x \rightarrow 8} f(x)$  if  $f(8) = 5$  and  $f$  is continuous at  $x = 8$ ?
- Sketch the graph of a function  $y = T(x)$  that is defined for  $-2 \leq x \leq 2$  and is continuous on  $[-2, 2)$  but is discontinuous on  $[-2, 2]$ .
- Use the formulas  $x^2$  and  $1/x$  to define a function  $y = W(x)$  for all  $x$  so it is continuous on  $(-\infty, 0]$  and  $(0, \infty)$  but not on  $(-\infty, \infty)$ . Draw its graph.
- (a) Use the formulas  $x+2$  and  $x+4$  to define a function  $y = V(x)$  for  $-2 \leq x \leq 3$  such that  $V(-2) = 0$ ,  $V(3) = 7$  but the equation  $V(x) = 4$  has no solutions. Draw its graph. (b) Relate the results of part (a) to the Intermediate Value Theorem.

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<sup>†</sup>In the published text the interactive solutions of these examples will be on an accompanying CD disk which can be run by any computer browser without using an internet connection.

**BASICS:**

- 5.<sup>0</sup> (a) Figure 23 shows the graph of a function  $Q$  defined for  $-2 \leq x \leq 6$ . At what values of  $x$  is  $Q$  continuous? (b) At what values is  $Q$  continuous from the left but not continuous? (c) At what values is  $Q$  continuous from the right but not continuous? (c) What are the largest intervals on which  $Q$  is continuous?

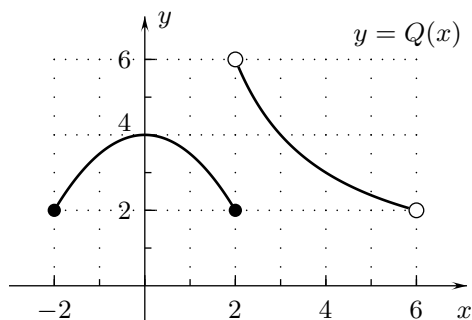


FIGURE 23

- 6.<sup>0</sup> On what intervals would  $Q$  of Figure 24 be continuous if its value at  $x = 2$  were changed to  $Q(2) = 6$ ?
- 7.<sup>0</sup> The function  $y = R(x)$  of Figure 24 is given by  $R(x) = \begin{cases} \cos x & \text{for } -20 \leq x \leq 0 \\ 2^{x/5} & \text{for } 0 < x \leq 15. \end{cases}$   
Use the formulas to explain why it is continuous on  $[-20, 15]$ .

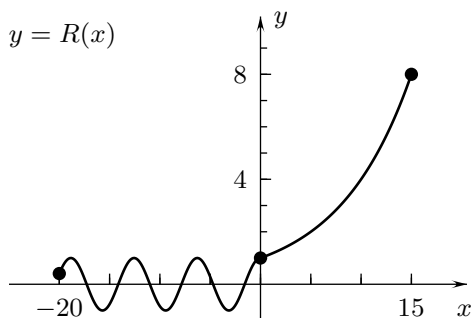


FIGURE 24

- 8.<sup>0</sup> What are the largest intervals on which  $y = \ln x + \sqrt{1-x}$  is continuous?
- 9.<sup>0</sup> What is the limit of  $y = \sqrt{x^2-1}$  as  $x$  tends to 1 from the right and why?
- 10.<sup>0</sup> What are  $\lim_{x \rightarrow 2^-} S(x)$  and  $\lim_{x \rightarrow 2^+} S(x)$  if  $S(x) = \begin{cases} \sin^2 x & \text{for } x < 2 \\ \cos^2 x & \text{for } x > 2 \end{cases}$ ?
11. (a) Generate  $y = x + 2x^2 + 3x^3 + 4x^4$  and  $y = 10,000$  in a suitable window to show graphically that the equation  $x + 2x^2 + 3x^3 + 4x^4 = 10,000$  has a solution in  $[0, 8]$ . (b) Use a calculator or computer to find the approximate solution.

In Exercises 13 through 16 (a) sketch the graphs of the functions and use the graphs to find (b) the points where the functions are continuous and (c) the largest intervals on which they are continuous.

$$13.^0 \quad f(x) = \begin{cases} 1 & \text{for } x < -1 \\ x^2 - 1 & \text{for } -1 \leq x \leq 1 \\ x - 1 & \text{for } x > 1 \end{cases}$$

$$15.^A \quad V(x) = \begin{cases} x & \text{for } x \leq -1 \\ 1/x & \text{for } -1 < x < 0 \\ 1/x & \text{for } 0 < x < 1 \\ x & \text{for } x \geq 1 \end{cases}$$

$$14.^0 \quad U(x) = \begin{cases} 1 & \text{for } x < -\pi \\ \cos x & \text{for } -\pi \leq x \leq 2\pi \\ 1 & \text{for } x > 2\pi \end{cases}$$

$$16. \quad Q(x) = \begin{cases} x^2 - 3 & \text{for } x \leq -1 \\ 2x^3 & \text{for } -1 < x < 1 \\ x^2 - 3 & \text{for } x \geq 1 \end{cases}$$

In Exercises 17 through 25 give the largest intervals on which the functions are continuous.

17.<sup>0</sup>  $y = \ln(x^2 - 1)$

21.  $y = \sin(\sqrt{x})$

18.<sup>A</sup>  $y = \frac{8}{e^x + 4}$

22.  $y = \frac{\cos x}{\ln x}$

19.  $y = \frac{\sqrt{x}}{x-1}$

23.<sup>A</sup>  $y = \sqrt{\ln x}$

20.<sup>A</sup>  $y = 1 + \frac{1}{x+1} + \frac{1}{x} - \frac{1}{x-1}$

24.  $y = \frac{e^x}{e^x - 1}$

25.  $y = \ln(x - 4)$

26. (a) Sketch the graph of

$$Q(x) = \begin{cases} 2/x & \text{for } x < -1 \\ 2x^3 & \text{for } -1 \leq x \leq 1 \\ 2/x & \text{for } x > 1. \end{cases}$$

(b) Find  $Q(-1)$ ,  $\lim_{x \rightarrow -1} Q(x)$ ,  $Q(0)$ ,  $\lim_{x \rightarrow 0} Q(x)$ ,  $Q(1)$ , and  $\lim_{x \rightarrow 1} Q(x)$ . (c) What are the greatest and least values of  $Q(x)$  for all  $x$ ? (d) Use the graph to find the three solutions  $x$  of  $Q(x) = 2x$ . (e) At what values of  $x$  is  $Q$  continuous?

27.<sup>0</sup> Without drawing the graph, determine whether  $y = g(x)$  is continuous (a) at  $x = 2$  and (b) at  $x = 10$ , where

$$g(x) = \begin{cases} x^3 + x & \text{for } x < 2 \\ 10 & \text{for } 2 \leq x \leq 10 \\ 1 + \sqrt{x+7} & \text{for } x > 10. \end{cases}$$

28.<sup>A</sup> At what values of  $x$  is  $y = f(x)$  discontinuous if

$$f(x) = \begin{cases} x^3 - x^4 + 2 & \text{for } x \leq -1 \\ 2x^5 + x + 3 & \text{for } -1 < x < 1 \\ 7 & \text{for } x \geq 1? \end{cases}$$

29.<sup>0</sup> What is the limit of  $y = \sqrt{x} + \frac{6}{x+2}$  as  $x \rightarrow 4$  and why?

30.<sup>A</sup> What is  $\lim_{x \rightarrow 1^-} f(x)$  for  $f(x) = 5 + \sqrt{1-x}$  and why?

31. What is the limit of  $g(x) = \cos(\sqrt{x}) + e^{\sqrt{x}}$  as  $x \rightarrow 0^+$  and why?

32. Find  $\lim_{x \rightarrow 2} S(x)$ , where

$$S(x) = \begin{cases} 5x + \sqrt{2-x} & \text{for } x < 2 \\ 14 & \text{for } x = 2 \\ x^3 + 2 & \text{for } x > 2. \end{cases}$$

**EXPLORATION:**

In Exercises 33 through 36 give the largest intervals on which the functions are continuous.

**33.<sup>A</sup>**  $y = \ln(\sqrt{x^2 - 4})$

**35.**  $y = \cos^{-1}(e^x)$

**34.<sup>A</sup>**  $y = \sqrt{\sin^{-1} x}$

**36.**  $y = \tan^{-1}(x^{9/8})$

**37.<sup>A</sup>** (a) Sketch the graph of

$$\Lambda(x) = \begin{cases} 2 + 2x & \text{for } -1 \leq x < 0 \\ 3 & \text{for } x = 0 \\ 2 - 2x & \text{for } 0 < x \leq 1. \end{cases}$$

(b) At what values of  $x$  is  $\Lambda(x)$  continuous? (c) For what values of  $x$  is  $\Lambda(x) \geq 2$ ? (d) Does  $\Lambda(x)$  increase or decrease as  $x$  increases from  $-1$  to  $0$ ? (e) Does  $\Lambda(x)$  increase or decrease as  $x$  increases from  $0$  to  $1$ ? (f) What one change in the definition of  $\Lambda(x)$  would make it continuous at  $x = 0$ ?

**38.<sup>A</sup>** (a) Sketch the graph of

$$M(x) = \begin{cases} 1/x & \text{for } x < 0 \\ x + 1 & \text{for } 0 \leq x \leq 2 \\ 3 & \text{for } x > 2. \end{cases}$$

(b) At what values of  $x$  is  $M$  continuous? For what values of  $k$  (i.e., in what  $k$ -intervals) does the equation  $M(x) = k$  have (c) one solution  $x$ , (d) an infinite number of solutions, and (e) no solutions? (f) Use the graph to find the two solutions of  $M(x) = x$ .

**39.** (a) Sketch the graph of

$$A(x) = \begin{cases} -2 - x^2 & \text{for } x < 0 \\ 0 & \text{for } x = 0 \\ 2 + x^2 & \text{for } x > 0. \end{cases}$$

(b) At what values of  $x$  is  $y = A(x)$  discontinuous? (c) For what values of  $k$  does the equation  $A(x) = k$  have a solution? (d) For what values of  $x$  is  $A(x) \leq A(0)$ ? (e) Does  $A(x)$  increase or decrease as  $x$  increases?

**40.<sup>A</sup>** Find the constant  $k$  such that  $Z$  is continuous at  $x = 1$ , where

$$Z(x) = \begin{cases} \sin^{-1} x & \text{for } -1 \leq x < 1 \\ k & \text{for } x = 1 \\ 2 \tan^{-1} x & \text{for } x > 1. \end{cases}$$

**41.** (a) For what two values of  $a$  does  $\lim_{x \rightarrow 5} Y(x)$  exist if

$$Y(x) = \begin{cases} a/x & \text{for } x < 5 \\ -1 & \text{for } x = 5 \\ x/a & \text{for } x > 5? \end{cases}$$

(b) For what one value of  $a$  is  $y = Y(x)$  continuous at  $x = 5$ ?

**42.<sup>A</sup>** Find constants  $a$  and  $b$  such that  $y = g(x)$  is continuous for all  $x$ , where

$$g(x) = \begin{cases} x^3 & \text{for } x < -1 \\ ax + b & \text{for } -1 \leq x < 1 \\ x^2 + 2 & \text{for } x \geq 1. \end{cases}$$

- 43.<sup>O</sup>** (a) Use the Intermediate Value Theorem to show that  $e^x - x^3 = 8$  has at least one solution in  $[-3, 3]$ . **(b)** Use a calculator or computer to find an approximate solution.
- 44.<sup>O</sup>** (a) Use the Intermediate-Value Theorem to show that the equation  $\frac{6 + x^3 - 2x}{x + 3} = 1$  has at least one solution. **(b)** Find the approximate value of one solution with a calculator or computer.
- 45.<sup>A</sup>** (a) Use the Intermediate-Value Theorem to show that the equation  $12x - 7x^2 + x^3 = 4$  has at least one solution in  $[0, 5]$ . **(b)** Use a calculator or computer to find approximate values of all solutions in  $[0, 5]$ .
- 46.** (a) Use the Intermediate-Value Theorem to show that the equation  $x + \tan^{-1} x = 10$  has at least one solution. **(b)** Find the approximate value of the solution with a calculator or computer.
- 47.<sup>A</sup>** (a) Sketch the graph of

$$W(x) = \begin{cases} -1 - x & \text{for } -2 \leq x < -1 \\ 1 + x & \text{for } -1 \leq x < 0 \\ 1 - x & \text{for } 0 \leq x < 1 \\ -1 + x & \text{for } 1 \leq x \leq 2. \end{cases}$$

- (b) Are the hypotheses of the Intermediate-Value Theorem satisfied for  $y = W(x)$  on the interval  $[-1, 2]$  with  $k = \frac{1}{2}$ ? **(c)** Does the conclusion hold in this case?
- 48.** (a) Sketch the graph of

$$C(x) = \begin{cases} -1 & \text{for } x < -1 \\ x^2 & \text{for } -1 \leq x \leq 1 \\ 1 & \text{for } x > 1. \end{cases}$$

- (b) Are the hypotheses of the Intermediate-Value Theorem satisfied for  $y = C(x)$  on the interval  $[-2, 2]$  with  $k = \frac{1}{2}$ ? **(c)** Does the conclusion hold in this case?
- 49.** (a) Sketch the graph of

$$f(x) = \begin{cases} x & \text{for } x < 0 \\ x + 1 & \text{for } x \geq 0. \end{cases}$$

- (b) Are the hypotheses of the Intermediate Value Theorem satisfied for  $y = f(x)$  on the interval  $[-1, 1]$  with  $k = \frac{1}{2}$ ? **(c)** Does the conclusion hold in this case?
- 50.** (a) Sketch the graph of

$$g(x) = \begin{cases} 3/x & \text{for } 0 < x \leq 2 \\ -2 + x & \text{for } 2 < x \leq 4. \end{cases}$$

- (b) Are the hypotheses of the Intermediate-Value Theorem satisfied for  $y = g(x)$  with  $k = 1$  on the interval  $[1, 4]$ ? **(c)** Does the conclusion hold in this case?

Use the Intermediate Value Theorem in Exercises 51 through 54. State the continuity assumptions that you make in your mathematical models.

- 51.<sup>A</sup>** A sprinter starts from rest and runs around a circular track. He stops when he gets back to the start. Show that at least once during that lap he must have had the same speed at diametrically opposite points.
- 52.** A rock climber starts to climb a mountain at 8:00 AM on a Saturday and gets to the top at 4:00 PM that afternoon. She camps on top and climbs back down on Sunday, starting at 8:00 AM and getting back to her starting point at 4:00 PM. Show that at some time of day on Sunday she was at the same elevation as she was at that time on Saturday.

- 53.<sup>A</sup>** Show that the equation  $ax^3 + bx^2 + cx + d = 0$  has at least one solution  $x$  for any choice of the constants  $a, b, c,$  and  $d$  with  $a \neq 0$ .
- 54.** Hank weighed one-half stone (seven pounds) at birth and ten stones (140 pounds) at age 15. **(a)** Prove that at some time his weight in stones equalled his age in years. **(b)** Did his weight in pounds have to equal his age in years at some time? **(c)** Did his weight in pounds have to equal his age at some time if his age were measured in months?
- 55.** Show that the equation  $F(x) = k$  has at least one solution for every  $k$  with  $0 \leq k \leq 200$ , where

$$F(x) = \begin{cases} x^3 & \text{for } 0 \leq x \leq 5 \\ 225 - \frac{2500}{x^2} & \text{for } 5 < x \leq 10. \end{cases}$$

- 56.** Give intervals  $[a, b]$  and numbers  $k$  such that for the function  $y = S(x)$  of Figure 25, **(a)** the hypotheses of the Intermediate-Value are satisfied, **(b)** the hypotheses of the Intermediate-Value Theorem are not satisfied and its conclusion does not hold, and **(c)** the hypotheses of the Intermediate-Value Theorem are not satisfied and its conclusion does hold.

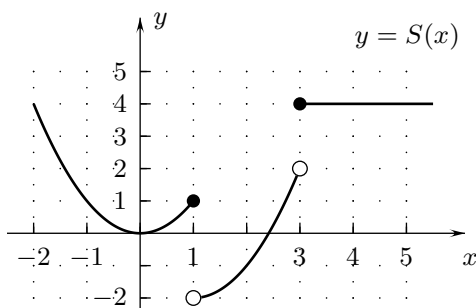
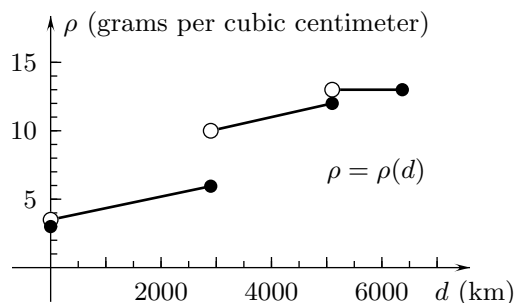


FIGURE 25

- 57.** Ten minutes after a hot dog vender opens her stand, she sells one hot dog. Fifteen minutes later she sells four hot dogs. After another half hour she sells three more. During the next thirty five minutes she does not sell any, so she goes home. Let  $H(t)$  be the number of hot dogs she sells during the first  $t$  minutes that her stand is open. Give formulas for  $H = H(t)$  and sketch its graph. Where is it continuous?
- 58.** A model rocket is fired vertically from the ground at time  $t = 0$  (seconds). It rises at the constant rate of 50 feet per second until  $t = 2$ , when it runs out of fuel, so that is is  $h(t) = 50t$  feet above the ground at time  $t$  for  $0 \leq t \leq 2$ . Under the assumption that there is no air resistance, then the rocket is  $h(t) = -16t^2 + 114t + C$  feet above the ground from  $t = 2$  until it hits the ground with a constant  $C$ . **(a)** Explain why it is reasonable to assume that  $h$  is continuous at  $t = 2$  and use this assumption to find  $C$ . **(b)** Sketch the graph of  $h$ . **(c)** Use the quadratic formula to give an exact expression for the time when the rocket hits the ground.

- 59.** Figure 26 shows the graph of the density of the earth  $\rho = \rho(d)$  as a function of the depth  $d$  (kilometers) below the surface.<sup>(1)</sup> The dot on the  $\rho$ -axis gives the density in the earth's crust, which is about 15 kilometers thick. The line on the left gives the density in the mantle, which extends to 2,900 kilometers. The two lines on the right give the density in the outer liquid core, which extends from 2,900 kilometers to 5,100 kilometers; and in the inner core, which extends to the center of the earth at a depth of 6,370 kilometers. **(a)** What is the advantage of representing the density of the earth with the discontinuous function  $\rho = \rho(d)$ ? **(b)** In what ways is this model unrealistic?

FIGURE 26



**(End of Section 1.3)**

<sup>(1)</sup>Data from *Encyclopædia Britannica*, Micropædia, Vol, III, p. 749, Chicago: Encyclopædia Britannica, Inc., 1974.