Section 1.4

(1/3/08)

Limits involving infinity

OVERVIEW: *In later chapters we will need notation and terminology to describe the behavior of functions in cases where the variable or the value of the function becomes large. We say that x or y* TENDS TO ∞ *if it becomes an arbitrarily large positive number and that x or y* TENDS TO $-\infty$ *if it becomes an arbitrarily large negative number.*[†] *These concepts are the basis of the definitions of several types of limits that we discuss in this section.*

Topics:

- ***Infinite limits as*** $\mathbf{x} \to \pm\infty$
- ***Finite limits as*** $\mathbf{x} \to \pm\infty$
- ***One-sided and two-sided infinite limits***
- ***Infinite limits of transcendental functions***

Infinite limits as $\mathbf{x} \to \pm\infty$

Imagine a point that moves on the curve $y = x^3$ in Figure 1. As the x-coordinate of the point increases through all positive values, the point moves to the right and rises higher and higher, so that it is eventually above any horizontal line, no matter how high it is. We say that x^3 TENDS TO ∞ AS x TENDS TO ∞ and write

$$\lim_{x\to\infty} x^3 = \infty.$$

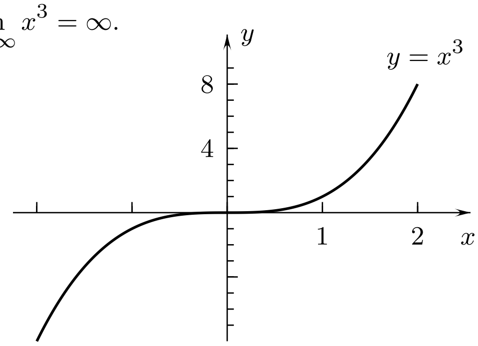


FIGURE 1

Similarly, as the x-coordinate of the point decreases through all negative values, the point moves to the left and drops lower and lower so that it is eventually beneath any horizontal line, regardless how low it is. We say that x^3 TENDS TO $-\infty$ AS x TENDS TO $-\infty$, and we write

$$\lim_{x\to-\infty} x^3 = -\infty.$$

The function $y = x^3$ illustrates the first and fourth parts of the following definition.

Definition 1 (Infinite limits as $\mathbf{x}$ ***tends to*** $\pm\infty$***)***

(a) $\lim_{x\to\infty} f(x) = \infty$ *if $f(x)$ is an arbitrarily large positive number for all sufficiently large positive x.*

(b) $\lim_{x\to\infty} f(x) = -\infty$ *if $f(x)$ is an arbitrarily large negative number for all sufficiently large positive x.*

(c) $\lim_{x\to-\infty} f(x) = \infty$ *if $f(x)$ is an arbitrarily large positive number for all sufficiently large negative x.*

(d) $\lim_{x\to-\infty} f(x) = -\infty$ *if $f(x)$ is an arbitrarily large negative number for all sufficiently large negative x.*

[†] When we say that a negative number x or y is "large," we mean that its absolute value is large.

Parts (a) and (b) of this definition apply only if f is defined on an interval (a, ∞) for some number a, and parts (c) and (d) apply only if f is defined on $(-\infty, b)$ for some b.

We can often determine the types of limits described in Definition 1 from the graphs of the functions, as in the next example.

Example 1 What are $\lim_{x\to\infty} x^2$ and $\lim_{x\to-\infty} x^2$?

SOLUTION The graph in Figure 2 shows that $\lim_{x\to\infty} x^2 = \infty$ and $\lim_{x\to-\infty} x^2 = \infty$. □

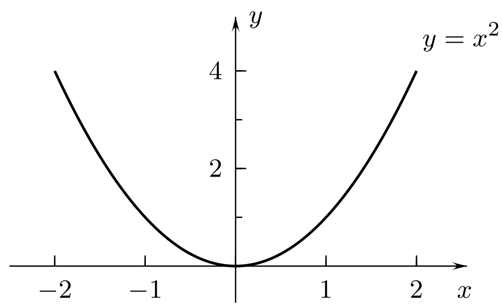


FIGURE 2

The next example illustrates a basic principle: any polynomial has the same limits as $x \to \infty$ and as $x \to -\infty$ as its term involving the highest power of x.

Example 2 Find $\lim_{x\to\infty}(2x^4 - 11x^3)$ and $\lim_{x\to-\infty}(2x^4 - 11x^3)$.

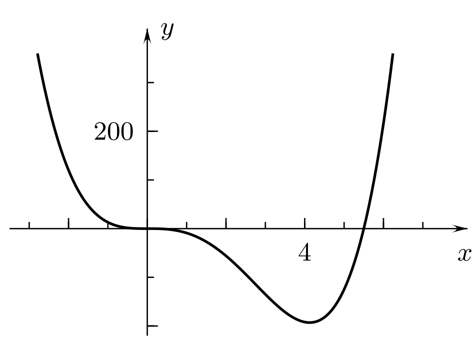
SOLUTION We can expect that the limits as $x \to \pm\infty$ of $2x^4 - 11x^3$ will be those of its highest degree term $2x^4$, so that

$$\begin{aligned} \lim_{x\to\infty}(2x^4 - 11x^3) &= \lim_{x\to\infty}(2x^4) = \infty \\ \lim_{x\to-\infty}(2x^4 - 11x^3) &= \lim_{x\to-\infty}(2x^4) = \infty. \end{aligned} \tag{1}$$

To verify these conclusions, we factor out the highest power of $2x^4 - 11x^3$ by writing for $x \neq 0$,

$$2x^4 - 11x^3 = x^4\left(2 - \frac{11}{x}\right).$$

This quantity tends to ∞ as $x \to \infty$ and as $x \to -\infty$ because $2 - 11/x$ tends to 2 and x^4 tends to ∞. Properties **(1)** of the function $y = 2x^4 - 11x^3$ can also be seen from its graph in Figure 3. □



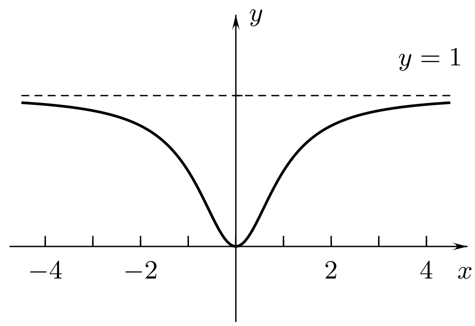
$y = 2x^4 - 11x^3$

FIGURE 3

Finite limits as $\mathbf{x} \to \pm\infty$

The function $y = 1 - 1/(1+x^2)$ of Figure 4 has a different sort of behavior for large positive and large negative x. Because $1/(1+x^2)$ is very small for large positive or negative x, the value $1 - 1/(1+x^2)$ approaches 1 and the graph approaches the horizontal line $y = 1$ as $x \to \infty$ and as $x \to -\infty$. We say that the LIMIT of $1 - 1/(1+x^2)$ as x tends to ∞ or to $-\infty$ is 1, and we write

$$\lim_{x\to\infty}\left(1 - \frac{1}{1+x^2}\right) = 1 \text{ and } \lim_{x\to-\infty}\left(1 - \frac{1}{1+x^2}\right) = 1.$$



$y = 1 - \dfrac{1}{1+x^2}$

FIGURE 4

Here is a general definition of this type of limit:

Definition 2 (Finite limits as $\mathbf{x}$ ***tends to*** $\pm\infty$***)***

(a) $\lim\limits_{x\to\infty} f(x) = L$ *with a number* L *if* $f(x)$ *is arbitrarily close to* L *for all sufficiently large positive* x.

(b) $\lim\limits_{x\to-\infty} f(x) = L$ *with a number* L *if* $f(x)$ *is arbitrarily close to* L *for all sufficiently large negative* x.

If $\lim\limits_{x\to\infty} f(x) = L$ or $\lim\limits_{x\to-\infty} f(x) = L$, then the line $y = L$ is a HORIZONTAL ASYMPTOTE of the graph. The line $y = 1$, for example, is a horizontal asymptote of the graph in Figure 4.

Example 3 Calculate the values of $f(x) = \dfrac{3-2x^4}{1+x^2+x^4}$ at $x = \pm 100, \pm 1000$, and $\pm$ 10,000 and use the results to predict the limits of $\dfrac{3-2x^4}{1+x^2+x^4}$ as $x \to \infty$ and as $x \to -\infty$.

SOLUTION The values in the table below suggest that $\lim\limits_{x\to\infty} \dfrac{3-2x^4}{1+x^2+x^4} = -2$ and $\lim\limits_{x\to\infty} \dfrac{3-2x^4}{1+x^2+x^4} = -2$. □

x	−10,000	−1000	−100	100	1000	10,000
$\dfrac{3-2x^4}{1+x^2+x^4}$	−1.99999998	−1.99999800	−1.99979997	−1.99979997	−1.99999800	−1.99999998

The limits as $x \to \pm\infty$ of a quotient of polynomials or other linear combinations of powers are the limits of the quotient of the highest-degree terms. If the numerator and denominator have the same degree, as in Example 3, this can be verified by dividing the numerator and denominator by x^n, where n is the degree of the numerator and denominator.

Example 4 Find $\lim\limits_{x\to\infty} \dfrac{3-2x^4}{1+x^2+x^4}$ and $\lim\limits_{x\to-\infty} \dfrac{3-2x^4}{1+x^2+x^4}$.

SOLUTION Since the term of highest degree in the numerator is $-2x^4$ and the term of highest degree in the denominator is x^4, we can anticipate that

$$\lim_{x\to\pm\infty} \frac{3-2x^4}{1+x^2+x^4} = \lim_{x\to\pm\infty} \frac{-2x^4}{x^4} = \lim_{x\to\pm\infty} (-2) = -2.$$

To verify this, we divide the numerator and denominator of the given function by x^4 to obtain for $x \neq 0$,

$$\frac{3-2x^4}{1+x^2+x^4} = \frac{\dfrac{3}{x^4} - 2}{\dfrac{1}{x^4} + \dfrac{1}{x^2} + 1}.$$

Then, since $3/x^4, 1/x^3$, and $1/x$ tend to 0 as $x \to \pm\infty$,

$$\lim_{x\to\pm\infty} \frac{3-2x^4}{1+x^2+x^4} = \frac{-2}{1} = -2.$$

The line $y = -2$ is a horizontal asymptote of the graph (Figure 5). □

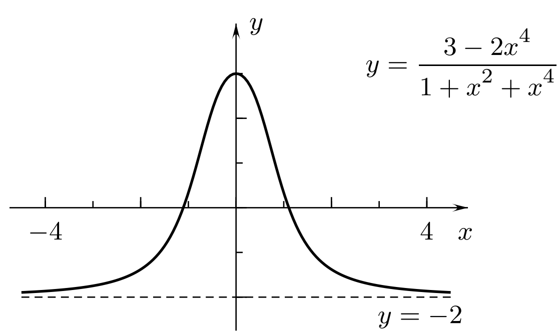


FIGURE 5

If the numerator and denominator of a quotient of polynomials are of different degrees, we find the limits of the quotient as $x \to \pm\infty$ by dividing the numerator and denominator by x^n, where n is the lower of the two degrees.

Example 5 Find $\lim_{x\to\infty} \dfrac{3x+1}{x^2+1}$ and $\lim_{x\to-\infty} \dfrac{3x+1}{x^2+1}$.

SOLUTION Since the terms of highest degree in the numerator and denominator are $3x$ and x^2, respectively, we anticipate that

$$\lim_{x\to\infty} \frac{3x+1}{x^2+1} = \lim_{x\to\infty} \frac{3x}{x^2} = \lim_{x\to\infty} \frac{3}{x} = 0.$$

The numerator is of degree 1 and the denominator is of degree 2, so to verify this conclusion we divide both by $x = x^1$. We obtain

$$\lim_{x\to\infty} \frac{3x+1}{x^2+1} = \lim_{x\to\infty} \frac{3+x^{-1}}{x+x^{-1}} = 0.$$

The last expression tends to 0 as x tends to ∞ because the numerator $3+x^{-1}$ tends to 3 and the denominator $x+x^{-1}$ tends to ∞.

By the same reasoning,

$$\lim_{x\to-\infty} \frac{3x+1}{x^2+1} = \lim_{x\to-\infty} \frac{3+x^{-1}}{x+x^{-1}} = 0. \ \square$$

Example 6 What are $\lim_{x\to\infty} \dfrac{x^3-x}{x^2+1}$ and $\lim_{x\to-\infty} \dfrac{x^3-x}{x^2+1}$?

SOLUTION We anticipate that $y = \dfrac{x^3-x}{x^2+1}$ has the same limits as $x \to \pm\infty$ as $y = \dfrac{x^3}{x^2} = x$, so that it tends to ∞ as $x \to \infty$ and tends to $-\infty$ as $x \to -\infty$.

To verify this, we divide the numerator and denominator by x^2 to obtain

$$\frac{x^3-x}{x^2+1} = \frac{x-x^{-1}}{1+x^{-2}} \quad \text{for } x \neq 0.$$

Since x^{-1} and x^{-2} tend to 0 as $x \to \pm\infty$, the last equation shows that

$$\lim_{x\to\infty} \frac{x^3-x}{x^2+1} = \lim_{x\to\infty} \frac{x-x^{-1}}{1+x^{-2}} = \infty$$

$$\lim_{x\to-\infty} \frac{x^3-x}{x^2+1} = \lim_{x\to-\infty} \frac{x-x^{-1}}{1+x^{-2}} = -\infty. \ \square$$

One-sided and two-sided infinite limits

Figure 6 shows the graph of $y = 1/x$. Because its values become arbitrarily large positive numbers as x approaches 0 from the right and become arbitrarily large negative numbers as x approaches 0 from the left, we say that $y = 1/x$ TENDS TO ∞ as x tends to 0 from the right and TENDS TO $-\infty$ as x approaches 0 from the left. We write

$$\lim_{x\to 0^+} \frac{1}{x} = \infty \quad \text{and} \lim_{x\to 0^-} \frac{1}{x} = \infty.$$

In contrast, the values of $y = 1/x^2$ in Figure 7 become arbitrarily large positive numbers as x approaches 0 from either side, so we say that $y = 1/x^2$ tends to ∞ as x tends to 0. We write

$$\lim_{x\to 0} \frac{1}{x^2} = \infty.$$

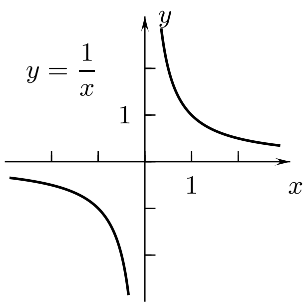


FIGURE 6

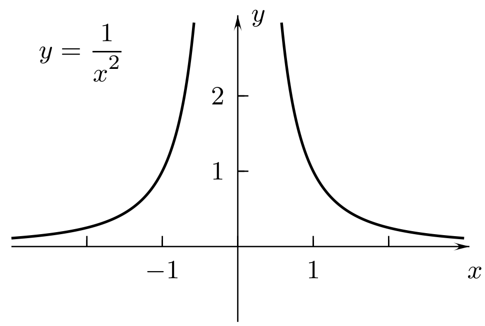


FIGURE 7

These functions illustrate the following definition:

Definition 3 (Infinite one-sided and two-sided limits)

(a) $\lim_{x \to a^+} f(x) = \infty$ *if* $f(x)$ *is an arbitrarily large positive number for all* $x > a$ *sufficiently close to* a.

(b) $\lim_{x \to a^-} f(x) = \infty$ *if* $f(x)$ *is an arbitrarily large positive number for all* $x < a$ *sufficiently close to* a.

(c) $\lim_{x \to a^+} f(x) = -\infty$ *if* $f(x)$ *is an arbitrarily large negative number for all* $x > a$ *sufficiently close to* a.

(d) $\lim_{x \to a^-} f(x) = -\infty$ *if* $f(x)$ *is an arbitrarily large negative number for all* $x < a$ *sufficiently close to* a.

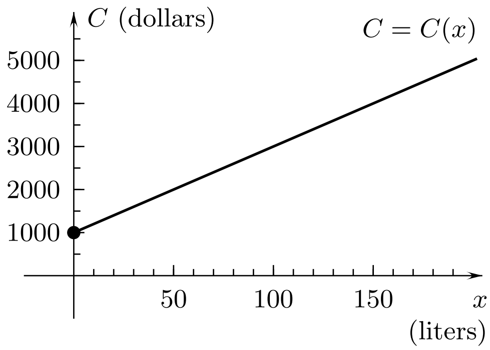
(e) $\lim_{x \to a} f(x) = \infty$ *if* $\lim_{x \to a^+} f(x) = \infty$ *and* $\lim_{x \to a^-} f(x) = \infty$.

(f) $\lim_{x \to a} f(x) = -\infty$ *if* $\lim_{x \to a^+} f(x) = -\infty$ *and* $\lim_{x \to a^-} f(x) = -\infty$.

The line $x = a$ is a VERTICAL ASYMPTOTE of $y = f(x)$ if $f(x) \to \infty$ or $f(x) \to -\infty$ as $x \to a^+$ or as $x \to a^-$. The y-axis ($x = 0$), for example, is the vertical asymptote of $y = 1/x$ in Figure 6 and of $y = 1/x^2$ in Figure 7.

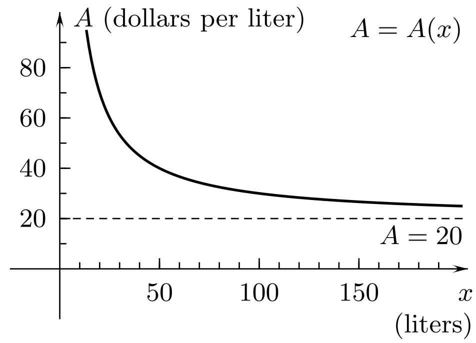
Example 7 A company produces a chemical at a cost of \$20 per liter plus a daily overhead (fixed cost) of \$1000. **(a)** What is the company's total cost $C = C(x)$ to produce x liters in a day? **(b)** What is the average total cost $A = A(x)$ per liter if x liters are produced in a day? **(c)** Find $\lim_{x\to\infty} A(x)$ and $\lim_{x\to 0^+} A(x)$ and explain the results.

SOLUTION **(a)** It costs [20 dollars per liter][x liters] $= 20x$ dollars to produce x liters, plus 1000 dollars overhead, so $C(x) = 1000 + 20x$ (dollars). The graph of this function is the line in Figure 8.



$C(x) = 1000 + 2x$

FIGURE 8



$A(x) = \dfrac{1000}{x} + 20$

FIGURE 9

(b) The average cost per liter is

$$A(x) = \frac{C(x)\ \text{dollars}}{x\ \text{liters}} = \frac{1000 + 20x}{x} = \frac{1000}{x} + 20\ \frac{\text{dollars}}{\text{liter}}. \tag{2}$$

(c) As can be seen from **(2)**

$$\lim_{x\to\infty} A(x) = \lim_{x\to\infty} \left[\frac{1000}{x} + 20\right] = 20\ \text{dollars} \tag{3}$$

and

$$\lim_{x\to 0^+} A(x) = \lim_{x\to\infty} \left[\frac{1000}{x} + 20\right] = \infty. \tag{4}$$

Consequently, the graph of the average cost in Figure 9 has the line $A = 20$ as a horizontal asymptote and the A-axis as a vertical asymptote.

The average cost per liter $A(x)$ is close to 20 for large x, as indicated by **(3)**, because if x is large, the overhead is spread out over a large number of liters of the chemical and the average cost is close to the cost of one liter with no overhead. The average cost is very large for very small x, as indicated by **(4)**, because for small x the overhead has to be covered by a small volume of the chemical. □

Example 8 Find $\lim_{x \to 0} (2x^{-2} + 4x^2 - x^3)$.

SOLUTION Since $4x^2 - x^3$ tends to 0 as $x \to 0$,

$$\lim_{x \to 0} (2x^{-2} + 4x^2 - x^3) = \lim_{x \to 0} 2x^{-2} = \lim_{x \to 0} \frac{2}{x^2} = \infty.$$

This property can be seen from the graph of $y = 2x^{-2} + 4x^2 - x^3$ in Figure 10. □

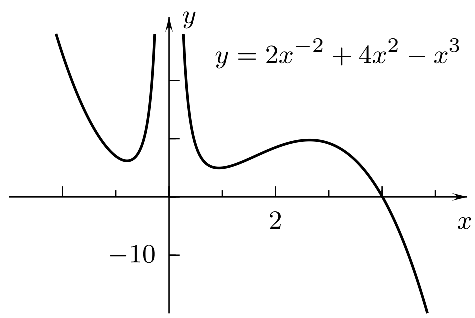


FIGURE 10

A quotient $y = p(x)/q(x)$ of polynomials has a vertical asymptote at any a such that $q(a) = 0$ and $p(a) \neq 0$ since the denominator tends to 0 and the numerator tends to the nonzero number $p(a)$ as $x \to a$. The behavior of the graph on both sides of the asymptote can be found by determining whether the function is positive or negative for x slightly greater than a and for x slightly less than a.†

Example 9 Find **(a)** $\lim_{x \to 1^+} \frac{x^3}{x-1}$, **(b)** $\lim_{x \to 1^-} \frac{x^3}{x-1}$, and **(c)** $\lim_{x \to 1} \frac{x^3}{x-1}$.

SOLUTION **(a)** The graph $y = \frac{x^3}{x-1}$ has a vertical asymptote at $x = 1$, where the denominator is 0 and the numerator is not zero. Because x^3 and $x-1$ are positive for $x > 1$, $y = \frac{x^3}{x-1}$ is positive for $x > 1$. Therefore, $\lim_{x \to 1^+} \frac{x^3}{x-1} = \infty$.

(b) On the other hand, x^3 is positive and $x-1$ is negative for $0 < x < 1$. Consequently, $y = \frac{x^3}{x-1}$ is negative for $0 < x < 1$. Consequently, $\lim_{x \to 1^-} \frac{x^3}{x-1} = -\infty$. These limits are shown by the graph of the function in Figure 11.

(c) The two-sided limit $\lim_{x \to 1^+} \frac{x^3}{x-1} = \infty$ is not defined because the one sided limits are different. □

† If $p(a) = 0$ and $q(a) = 0$, then a power of $x - a$ must be factored from the numerator and denominator and cancelled so that the new numerator and/or the new denominator are not zero at a before you can determine whether the graph has a vertical asymptote at $x = a$.

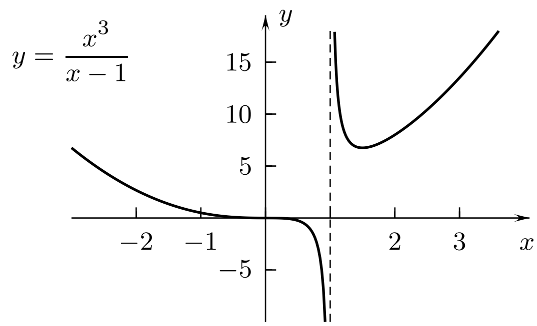


FIGURE 11

Infinite limits of transcendental functions

The exponential function $y = e^x$ tends to ∞ as $x \to \infty$ tend to 0 as $x \to -\infty$ (Figure 12). The logarithm $y = \ln x$ tends to $-\infty$ as $x \to 0^+$ and tend to ∞ as $x \to \infty$ (Figure 13).

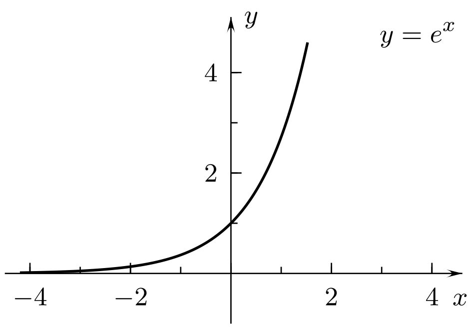


FIGURE 12

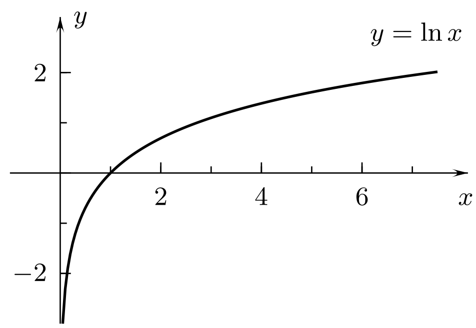


FIGURE 13

The next example deals with a function constructed from $y = e^x$.

Example 10 Figure 14 shows the curve $y = e^{1/x}$. Explain why **(a)** $\lim_{x\to\pm\infty} e^{1/x} = 1$, **(b)** $\lim_{x\to 0^+} e^{1/x} = \infty$, and **(c)** $\lim_{x\to 0^-} e^{1/x} = 0$.

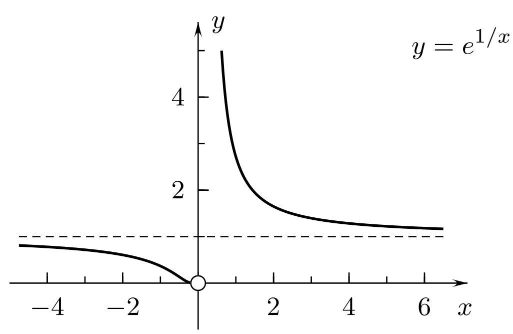


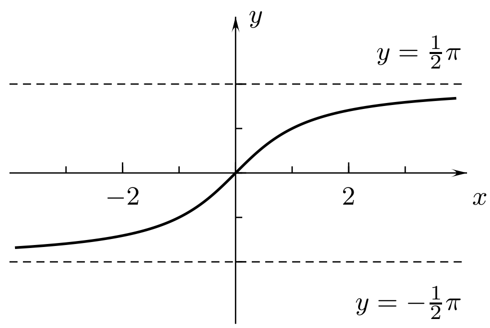
FIGURE 14

SOLUTION **(a)** $\lim_{x\to\pm\infty} e^{1/x} = 1$ because $z = 1/x \to 0$ as $x \to \pm\infty$ and $\lim_{z\to 0} e^z = e^0 = 1$.

(b) $\lim_{x\to 0^+} e^{1/x} = \infty$ because $z = 1/x \to \infty$ as $x \to 0^+$ and $\lim_{z\to\infty} e^z = \infty$.

(c) $\lim_{x\to 0^-} e^{1/x} = 0$ because $z = 1/x \to -\infty$ as $x \to 0^+$ and $\lim_{z\to -\infty} e^z = 0$. □

The trigonometric functions do not have limits as $x \to \pm\infty$ because they are periodic. The inverse sine and cosine functions do not have limits as $x \to \pm\infty$ because they are defined only for $-1 \le x \le 1$. The inverse tangent function, in contrast, tends to $\frac{1}{2}\pi$ as $x \to \infty$ and tends to $-\frac{1}{2}\pi$ as $x \to -\infty$ (Figure 15).



$y = \tan^{-1} x$

FIGURE 15

The tangent, cotangent, secant, and cosecant functions have infinite one-sided limits at the points where their denominators are zero. Other functions constructed from trigonometric functions can also have infinite limits, as in the next example.

Example 11 Find **(a)** $\lim_{x\to 0^+} \dfrac{\cos x}{x}$ and **(b)** $\lim_{x\to 0^-} \dfrac{\cos x}{x}$.

SOLUTION **(a)** For small positive x, $\cos x$ is close to $\cos(0) = 1$ and $1/x$ is a large positive number, so $\lim_{x\to 0^+} \dfrac{\cos x}{x} = \infty$.

(b) For small negative x, $\cos x$ is close to $\cos(0) = 1$ and $1/x$ is a large negative number, so $\lim_{x\to 0^-} \dfrac{\cos x}{x} = -\infty$. These properties can be seen from the graph in Figure 16. □

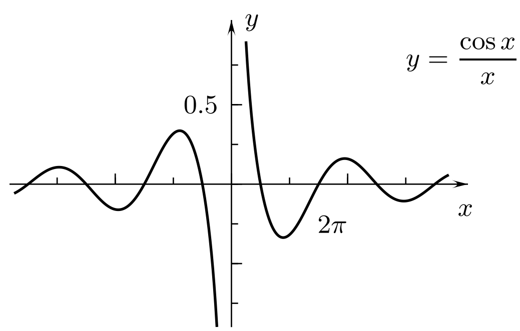


FIGURE 16

Example 12 Find **(a)** $\lim_{x\to\infty} \tan^{-1}(e^x)$ and **(b)** $\lim_{x\to-\infty} \tan^{-1}(e^x)$.

SOLUTION **(a)** $\tan^{-1}(e^x) \to \frac{1}{2}\pi$ as $x \to \infty$ because $e^x \to \infty$ as $x \to \infty$ and $\tan^{-1} y \to \frac{1}{2}\pi$ as $y \to \infty$.
(b) $\tan^{-1}(e^x) \to 0$ as $x \to -\infty$ because $e^x \to 0$ as $x \to -\infty$ and $\tan^{-1} y \to 0$ as $y \to 0$ (Figure 17)

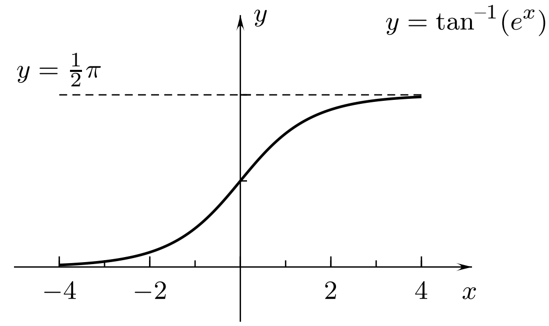


FIGURE 17

Interactive Examples 1.4

Interactive solutions are on the web page http//www.math.ucsd.edu/˜ashenk/.[†]

1. Find **(a)** $\lim_{x\to\infty} f(x)$ and **(b)** $\lim_{x\to-\infty} f(x)$ for $f(x) = 3x^2 - x^4$. $^{\mathsf{C}}$**(c)** Generate the graph of f in the window $-2 \le x \le 2, -3 \le y \le 3$ as a partial check of your answers.

2. What are the limits of $g(x) = \dfrac{x^3}{x^2 - 2x + 1}$ **(a)** as $x \to \infty$ and **(b)** as $x \to -\infty$? $^{\mathsf{C}}$**(c)** Generate the graph of g in the window $-6 \le x \le 9, -5 \le y \le 15$ as a partial check of your answers.

3. Find **(a)** $\lim_{x\to 2^+} \dfrac{1-4x}{x-2}$ and **(b)** $\lim_{x\to 2^-} \dfrac{1-4x}{x-2}$.

4. Find $\lim_{x\to -1} \dfrac{x^3}{(x+1)^2}$.

5. What are the limits of $y = \dfrac{1}{\ln x}$ **(a)** as $x \to 0^+$, **(b)** as $x \to 1^-$ **(c)** as $x \to 1^+$ and **(d)** as $x \to \infty$? (See Figure 18.)

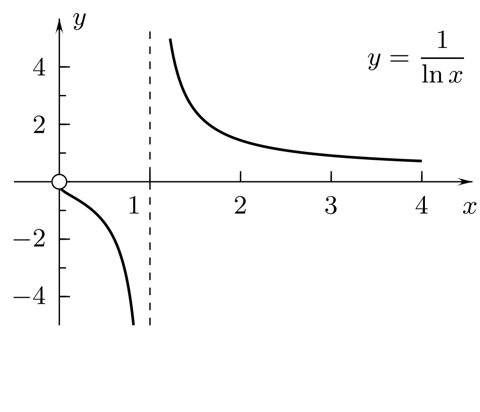


FIGURE 18

[†]In the published text the interactive solutions of these examples will be on an accompanying CD disk which can be run by any computer browser without using an internet connection.

Exercises 1.4

[A]Answer provided. [O]Outline of solution provided. [C]Graphing calculator or computer required.

CONCEPTS:

1. What is $\lim_{x\to\infty}[f(x) - 200]$ if $\lim_{x\to\infty} f(x) = \infty$?

2. What is $\lim_{x\to\infty}[f(x) - 200]$ if $\lim_{x\to\infty} f(x) = 0$?

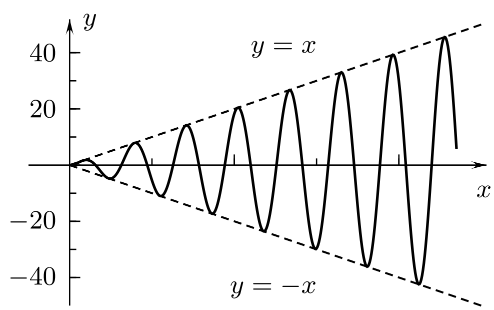
3. What is $\lim_{x\to\infty}[200 - f(x)]$ if $\lim_{x\to\infty} f(x) = -\infty$?

4. What is $\lim_{x\to\infty}\frac{1}{f(x)}$ if $\lim_{x\to\infty} f(x) = 0$ and $f(x) > 0$ for $x > 1000$?

5. What is $\lim_{x\to 0^+}\frac{1}{f(x)}$ if $\lim_{x\to 0^+} f(x) = 0$ and $f(x) < 0$ for $0 < x < 0.0001$?

6. Suppose that $x = 1$ is an asymptote of $y = f(x)$. Why is $x = 1$ also an asympote of $y = [f(x)]^2$?

7. **(a)** Explain the shape of the graph $y = x\sin x$ in Figure 19. **(b)** Use Definitions 1a and 1b to explain why $y = x\sin x$ does not tend to ∞ or $-\infty$ as $x \to \infty$.



$y = x\sin x$

FIGURE 19

BASICS:

8.[O] Figure 20 shows the graph of the polynomial $p(x) = -2x^3 + 5x^2 + 2$ Use the formula to find **(a)** $\lim_{x\to\infty} p(x)$ and **(b)** $\lim_{x\to-\infty} p(x)$.

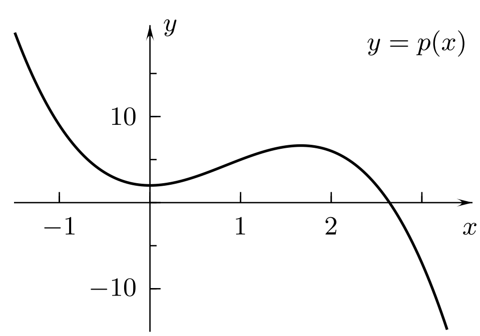


FIGURE 20

9.[O] Figure 21 shows the graph of $\dfrac{x^2}{(x-1)^2}$. Use the formula to find **(a)** $\lim_{x\to\infty} f(x)$, **(b)** $\lim_{x\to-\infty} f(x)$, and **(c)** $\lim_{x\to 1} f(x)$.

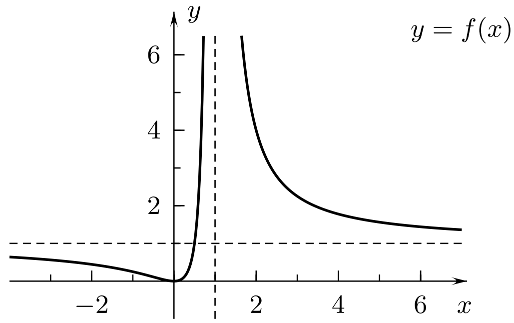


FIGURE 21

10.[O] What is $\lim_{x\to\infty} \dfrac{10\sqrt{x}-5}{2\sqrt{x}+4}$?

11.[O] Find $\lim_{x\to\infty} y(x)$ and $\lim_{x\to-\infty} y(x)$, where $y(x) = \dfrac{x-x^5}{1+x^4}$.

12.[O] What is $\lim_{x\to\infty} \dfrac{x}{1+x^6}$?

13.[O] Find **(a)** $\lim_{x\to 4^+} \dfrac{x}{(x-4)^3}$ and **(b)** $\lim_{x\to 4^-} \dfrac{x}{(x-4)^3}$.

In Exercises 14 through 21 use the formulas to find **(a)** $\lim_{x\to\infty} f(x)$ and **(b)** $\lim_{x\to-\infty} f(x)$.[C]**(c)** Generate the graphs in the given windows as a partial check of your answers.

14.[O] $f(x) = 4x^2 - x^3$ $(-2 \le x \le 5, -10 \le y \le 15)$

15.[O] $f(x) = x - \frac{1}{3}x^3$ $(-3 \le x \le 3, -2 \le y \le 2)$

16.[A] $f(x) = x + \frac{1}{3}x^3$ $(-2 \le x \le 2, -4 \le y \le 4)$

17. $f(x) = -3x + 2x^2 - \frac{1}{3}x^3$ $(-1 \le x \le 5, -3 \le y \le 3)$

18.[O] $f(x) = \dfrac{x^2-1}{x^2+1}$ $(-4 \le x \le 4, -1.5 \le y \le 1.5)$

19.[A] $f(x) = \dfrac{x^2}{x-2}$ $(-15 \le x \le 20, -15 \le y \le 20)$

20. $f(x) = \dfrac{x^2+1}{x^2-1}$ $(-4 \le x \le 4, -6 \le y \le 8)$

21. $f(x) = \dfrac{x^3}{1-x}$ $(-5 \le x \le 4, -20 \le y \le 20)$

In Exercises 22 through 25 use the formulas to find **(a)** $\lim_{x\to\infty} f(x)$, **(b)** $\lim_{x\to-\infty} f(x)$, **(c)** $\lim_{x\to 0^+} f(x)$, **(d)** $\lim_{x\to 0^-} f(x)$, and **(e)** $\lim_{x\to 0} f(x)$.[C]**(f)** Generate the graphs in the given windows as a partial check of your answers.

22.[O] $f(x) = 4 - x^2 + 1/x$ $(-3 \le x \le 3, -4 \le y \le 8)$

23. $f(x) = 2x + \dfrac{1}{x}$ $(-3 \le x \le 3, -8 \le y \le 8)$

24.[A] $f(x) = x^2 - \dfrac{1}{x^2}$ $(-3 \le x \le 3, -5 \le y \le 5)$

25. $f(x) = x^3 - \dfrac{1}{x}$ $(-3 \le x \le 3, -10 \le y \le 10)$

26.[O] Find $\lim\limits_{x \to 2} \dfrac{1-x}{(x-2)^2}$.

27.[A] Find $\lim\limits_{x \to 10^+} \dfrac{x+5}{x-10}$ and $\lim\limits_{x \to 10^-} \dfrac{x+5}{x-10}$.

28. Find $\lim\limits_{x \to 3} \dfrac{5-x^2}{(x-3)^2}$.

29.[A] What is $\lim\limits_{x \to \infty} \cos(e^{-x})$?

30. Find $\lim\limits_{x \to 0^+} \tan^{-1}(1/x)$.

EXPLORATION:

In Exercises 31 and 32 use the formulas to find **(a)** $\lim\limits_{x \to \infty} f(x)$, **(b)** $\lim\limits_{x \to -\infty} f(x)$, **(c)** $\lim\limits_{x \to 0^+} f(x)$, **(d)** $\lim\limits_{x \to 0^-} f(x)$, and **(e)** $\lim\limits_{x \to 0} f(x)$. [C]**(f)** Generate the graphs in the given windows as a partial check of your answers.

31.[O] $f(x) = (2 + 1/x)(3 - 5/x^2)$ $(-6 \le x \le 4, -6 \le y \le 8)$

32. $f(x) = \left(x^2 + \dfrac{1}{x^2}\right)\left(1 - x + x^4\right)$ $(-2 \le x \le 2,\ -5 \le y \le 25)$

33.[A] What is $\lim\limits_{x \to 0} \dfrac{2F(x)+1}{F(x)+2}$ if $\lim\limits_{x \to 0} F(x) = \infty$?

34. Find $\lim\limits_{x \to \infty} \dfrac{5\sqrt{x}}{(1+\sqrt[4]{x})^2}$.

35.[A] What are the aympototes of $y = \dfrac{e^x}{1+e^x}$?

36. Explain why $\lim\limits_{x \to (\pi/2)^-} \sec x = \infty$ **(a)** by using properties of $y = \cos x$ and **(b)** by using Figure 22 with the definition of $y = \sec x$ from trigonmetry.

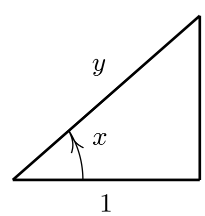


FIGURE 22

37. The curve $y = x - 4\tan^{-1} x$ in Figure 23 has two nonhorizontal, nonvertical asymptotes. What are they?

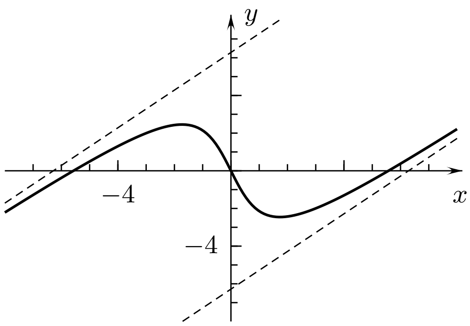


FIGURE 23

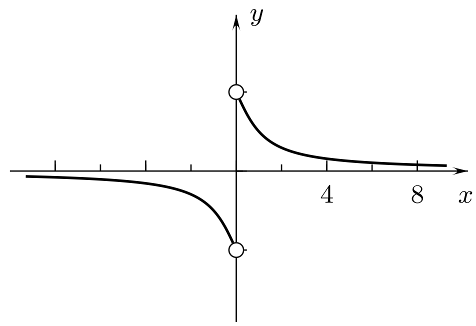


FIGURE 24

38. Find the limits of $y = \tan^{-1}(1/x)$ in Figure 24 as $x \to \pm\infty$ and as $x \to 0^{\pm}$.

39. Which of the curves **(a)** $y = \cos\left(\dfrac{1}{1+x^2}\right)$ **(b)** $y = \sin\left(\dfrac{1}{1+x^2}\right)$ is in Figure 25 and which is in Figure 26? Explain.

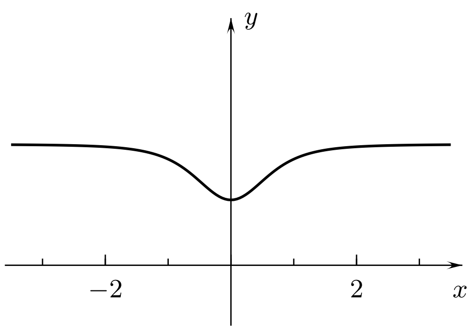


FIGURE 25

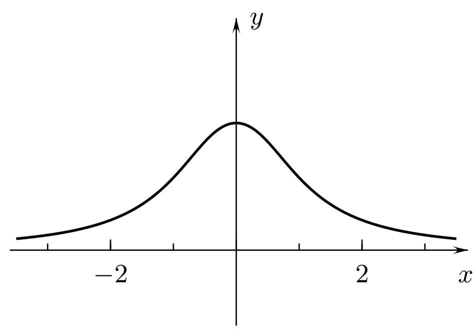


FIGURE 26

40. What are the asymptotes of $y = \dfrac{\ln x}{\ln x - 1}$?

41. What are the asymptotes of $y = \dfrac{1}{(\ln x)^2 - 10\ln x + 25}$?

42. Find $\displaystyle\lim_{x\to 2^+} \frac{x^3 - 2x^2}{x^2 - 4x + 4}$

43. What is the number C if $\displaystyle\lim_{x\to\infty} C\cos x$ is a number L?

(End of Section 1.4)