Section 2.2

(3/19/08)

Average rates of change

OVERVIEW: *This section is background for the definition of the derivative in the next section. The examples of average velocity and other average rates of change considered here illustrate the types of mathematical models that we will use later with derivatives, and the derivative will be defined as a limit of average rates of change.*

Topics:

- ***Average velocity***
- ***Other average rates of change***
- ***Estimating instantaneous velocities with average velocity***

Average velocity

Imagine that a pilot is flying a small airplane toward the west from an airport. As a mathematical model of her flight, we suppose that the plane is $s(t) = t^3 + 30t + 100$ miles from the airport t hours after noon (Figure 1). We begin by calculating the plane's average velocity during a particular period of time.

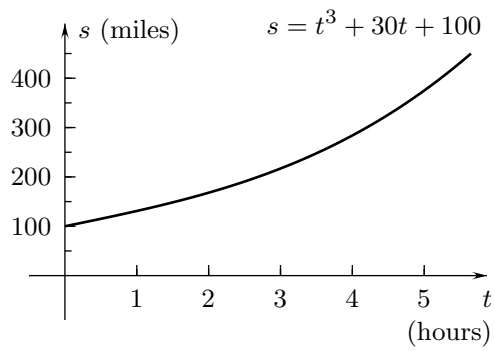


FIGURE 1

Example 1 What is the plane's average velocity from $t = 1$ to $t = 5$?

SOLUTION The plane's average velocity is equal to the distance it travels divided by the time taken. Since $s(1) = 131$ and $s(5) = 375$, its average velocity for $1 \le t \le 5$ is

$$\begin{aligned} \frac{\text{Distance traveled}}{\text{Time taken}} &= \frac{[s(5) - s(1)]\text{ miles}}{(5-1)\text{ hours}} = \frac{(375 - 131)\text{ miles}}{4\text{ hours}} \\ &= \frac{244\text{ miles}}{4\text{ hours}} = 61\ \frac{\text{miles}}{\text{hour}}. \ \square \end{aligned} \tag{1}$$

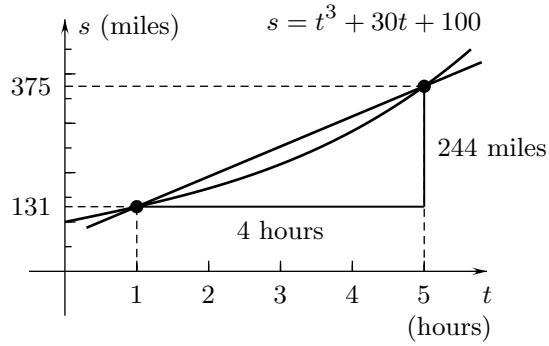
Notice that the average velocity **(1)** is the slope of the line in Figure 2 through the points at $t = 1$ and $t = 5$ on the graph of the plane's distance from the airport. The rise is the distance the plane travels, and the run the time it takes to go that distance. The line is called a SECANT LINE because it passes through two points on the curve.†

†The terminology "secant line" comes from the Latin word *secare*, meaning "to cut." It is used because secant lines generally cut across curves.

$$\begin{bmatrix}\text{Average}\\ \text{velocity}\end{bmatrix} = \frac{\text{Rise}}{\text{Run}}$$

$$= \frac{244 \text{ miles}}{4 \text{ hours}} = 61\ \frac{\text{miles}}{\text{hour}}$$

FIGURE 2



Any average velocity can be interpreted as the slope of a secant line:

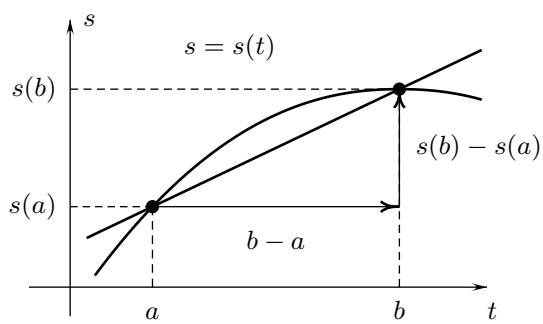
Definition 1 *If an object is at $s = s(t)$ on an s-axis at time t, then its* AVERAGE VELOCITY *in the positive s-direction between times a and b is its change in position, $s(b) - s(a)$, divided by the change in time, $b - a$. This ratio is the slope of the secant line through the points at $t = a$ and $t = b$ on the graph of $s(t)$ (Figure 3):*

$$\begin{aligned}[\text{Average velocity from time } a \text{ to time } b] &= \frac{\text{Change in position}}{\text{Change in time}} \\ &= \frac{s(b) - s(a)}{b - a} \quad (a \neq b).\end{aligned} \tag{2}$$

$$\begin{bmatrix}\text{Average}\\ \text{velocity}\end{bmatrix} = \frac{\text{Rise}}{\text{Run}}$$

$$= \frac{s(b) - s(a)}{b - a}$$

FIGURE 3



Other average rates of change

Average velocity is the AVERAGE RATE OF CHANGE of distance with respect to time. Consequently, Definition 1 is a special case of the following general definition of average rate of change.

Definition 2 *The* AVERAGE RATE OF CHANGE *of $y = f(x)$ with respect to x from $x = a$ to $x = b$ is*

$$\frac{\text{Change in } f(x)}{\text{Change in } x} = \frac{f(b) - f(a)}{b - a} \quad (a \neq b) \tag{3}$$

and equals the slope of the secant line through the points at $x = a$ and $x = b$ on the graph of f (Figure 4).

If x and $f(x)$ have units, then the units used for the average rate of change in Definition 2 are the units used for f divided by the units used for x.

$$\begin{bmatrix}\text{Average rate}\\ \text{of change}\end{bmatrix} = \frac{\text{Rise}}{\text{Run}} = \frac{f(b)-f(a)}{b-a}$$

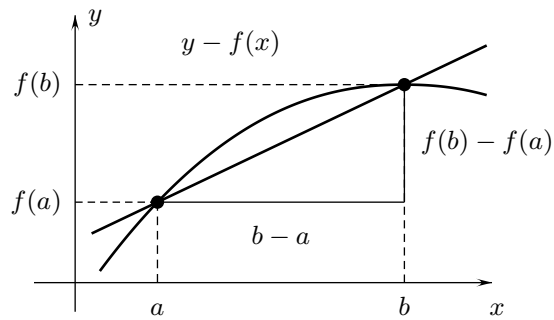


FIGURE 4

Example 2 Figure 5 shows the graph of the price $P = P(t)$ (dollars per barrel) of crude oil on the New York Mercantile Exchange with t measured in days and $t = 1$ on January 1, 2004. The price was 33.70 on January 5, 2004 ($t = 5$); 98.81 on August 30, 2005 ($t = 608$); and 71.95 on April 20, 2006 ($t = 841$). What was the average rate of change of the price with respect to time **(a)** from January 5, 2004 to August 30, 2005 and **(b)** from August 30, 2005 to April 20, 2006?

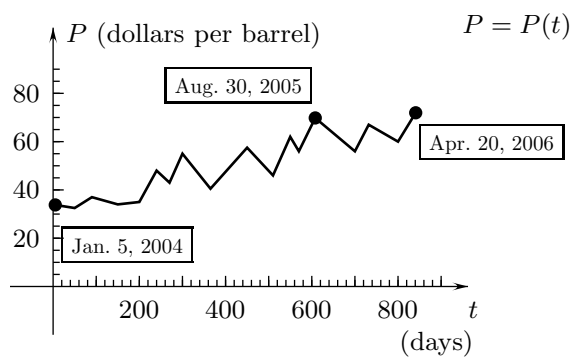


FIGURE 5

SOLUTION **(a)** The average rate of change of the price with respect to time from January 5, 2004 ($t = 5$) to August 30, 2005 ($t = 608$) was

$$\frac{P(608)-P(5)}{608-5} = \frac{69.81-33.78}{603} = \frac{36.03}{603} \doteq 0.06 \text{ dollars per barrel per day.}$$

(b) The average rate of change from August 30, 2005 ($t = 605$) to April 20, 2006 ($t = 841$) was

$$\frac{P(841)-P(608)}{841-608} = \frac{71.95-69.81}{233} = \frac{2.14}{233} \doteq 0.01 \text{ dollars per barrel per day.} \ \square$$

Example 3 What is the average rate of change of the width $w(A) = \sqrt[3]{A}$ (meters) of a cube with respect to its volume V (cubic meters) from $V = 1$ to $V = 8$?

SOLUTION The average rate of change is

$$\frac{w(8)-w(1)}{8-1} = \frac{\sqrt[3]{8}-\sqrt[3]{1}}{7} = \frac{2-1}{7} = \tfrac{1}{7} \text{ meters per cubic meter.} \ \square$$

Example 4 Bats are warm-blooded mammals whose body temperatures keep fairly constant when they are awake and active. When a bat is asleep in a cold place, however, it goes into a sort of hybernation and its metabolism (rate of energy expenditure) drops. The next table gives the metabolism r of a sleeping brown bat as a function of the air temperature T around it.[1] What is the average rate of change with respect to temperature of the bat's metabolism for $20 \le T \le 30$?

T = Temperature (°C)	0.5	2.0	10.0	20.0	30.0	37.0	41.5
r = Metabolic rate (Calories/hour)	5.4	1.4	3.4	19.0	96.0	134.0	200.0

SOLUTION Since $r(20) = 19.0$ Calories per hour and $r(30) = 96.0$ Calories per hour, the average rate of change of r with respect to T for $20 \le T \le 30$ is

$$\frac{r(30) - r(20)}{30 - 20} = \frac{96.0 - 19.0}{10} = 7.7 \text{ Calories per hour per degree. } \square$$

Example 5 What is the average rate of change of the bat's metabolism with respect to temperature for $0.5 \le T \le 2.0$? Interpret its sign.

SOLUTION The average rate of change is $\dfrac{r(2.0) - r(0.5)}{2.0 - 0.5} = \dfrac{1.4 - 5.4}{1.5} = -\frac{8}{3}$ Calories per hour per degree. It is negative because the bat's metabolism increases as the temperature drops from 2°C toward freezing (0°C). $\square$

If we have the graph of a function and not an exact formula for its values, we cannot find its exact average rates of change. We can only estimate them by estimating values of the function from the graph.

Example 6 Figure 6 shows the hours of sunshine $h = h(t)$ in Ft. Vermillion, Alberta, Canada as a function of the time of year t measured in months with $t = 0$ at the beginning of the year.[2] What is the approximate average rate of change of the hours of sunshine with respect to time from the beginning of April at $t = 3$ to the beginning of July at $t = 6$?

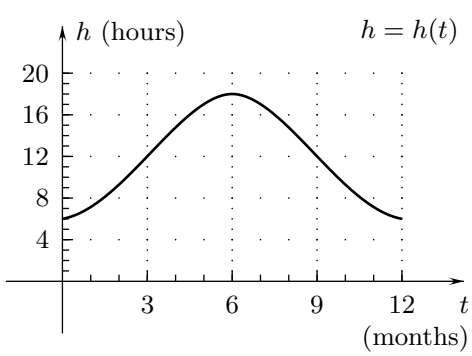


FIGURE 6

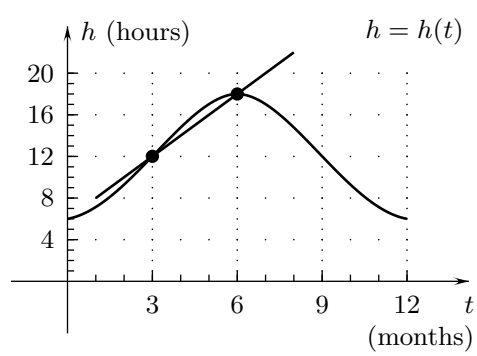


FIGURE 7

[1] Data adapted from *Listening in the Dark, The Acoustic Orientation of Bats and Men*, Donald R. Griffin, Cornell University Press, Ithaca, N.Y., 1986, p. 40.

[2] Data adapted from *Introduction to Physical Geography* by A. Strahler, Second Edition, New York NY: John Wiley & Sons, Inc., 1970, p. 172.

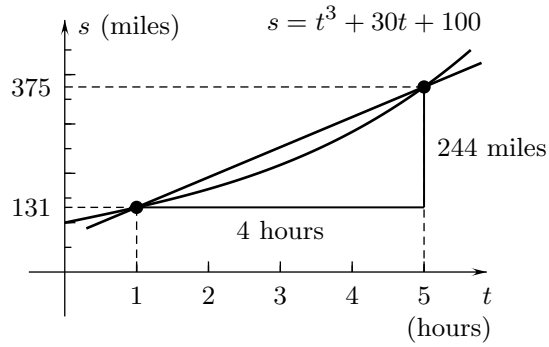
SOLUTION With the approximate values $h(3) \approx 12$ and $h(6) \approx 18$ from the graph, we find that the average rate of change of h with respect to t for $3 \le t \le 6$ is approximately

$$\frac{h(6)-h(3)}{6-3} = \frac{18-12}{3} = 2 \text{ hours per month.}$$

The corresponding secant line is shown in Figure 7. □

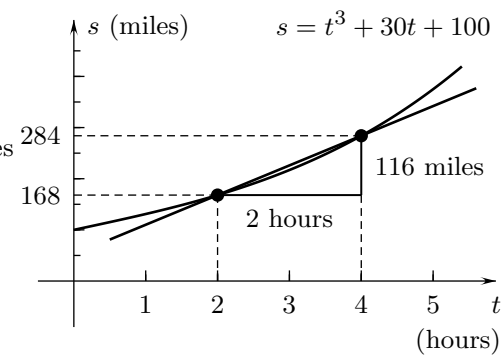
Estimating instantaneous velocity with average velocity

In Example 1 we saw that if an airplane is $s(t) = t^3 + 30t + 100$ miles west of an airport at time t (hours), then its average velocity toward the west for $1 \le t \le 5$ is 61 miles per hour, which is the slope of the secant line to the graph of $s = s(t)$ in Figure 8. The plane's average velocity for $2 \le t \le 4$ is $\frac{s(4)-s(2)}{4-2} = \frac{284-168}{2} = 58$ miles per hour, and is the slope of the secant line in Figure 9. These numbers differ considerably and neither is a very good estimate of how fast the plane is traveling at $t = 3$ because the function $s = s(t)$ is not linear and the time intervals considered are relatively long.



$$\begin{bmatrix}\text{Average}\\ \text{velocity}\end{bmatrix} = 61\ \frac{\text{miles}}{\text{hour}}$$

FIGURE 8



$$\begin{bmatrix}\text{Average}\\ \text{velocity}\end{bmatrix} = 58\ \frac{\text{miles}}{\text{hour}}$$

FIGURE 9

Figures 10 through 12 show the graph of the function $s = s(t)$ in time intervals of decreasing widths, centered at $t = 3$. Notice that as the time interval gets smaller, the graph looks more like a line. Because of this, we can obtain better estimates of the plane's velocity at $t = 3$ from its average velocity in the shorter time intervals.

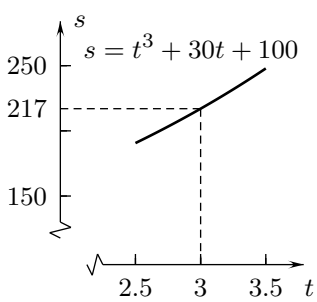


FIGURE 10

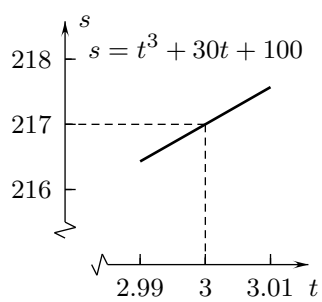


FIGURE 11

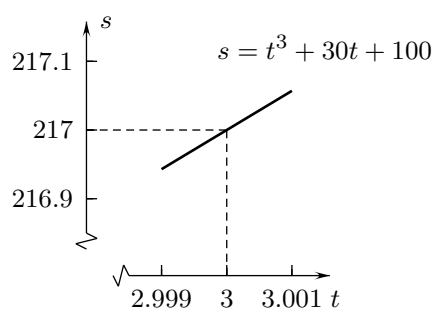


FIGURE 12

Example 7 Find the plane's average velocity in the time intervals **(a)** $2.5 \le t \le 3.5$, **(b)** $2.99 \le t \le 3.01$, and **(c)** $2.999 \le t \le 3.001$ that correspond to the graphs in Figures 10 through 12.

SOLUTION **(a)** The plane's average velocity for $2.5 \le t \le 3.5$ is

$$\frac{s(3.5) - s(2.5)}{3.5 - 2.5} = \frac{247.875 - 190.625}{1} = 57.25 \text{ miles per hour.}$$

(b) Its average velocity for $2.99 \le t \le 3.01$ is

$$\frac{s(3.01) - s(2.99)}{3.01 - 2.99} = \frac{217.570901 - 216.430899}{0.02} = 57.0001 \text{ miles per hour.}$$

(c) Its average velocity for $2.999 \le t \le 3.002$ is

$$\frac{s(3.001) - s(2.999)}{3.001 - 2.999} \doteq \frac{217.057009 - 216.94309}{0.002} = 57.000001 \text{ miles per hour.} \square$$

We can conclude from the results of Example 6 that the plane's velocity is approximately 57 miles per hour at $t = 3$. We refer to an object's velocity at a particular moment as its INSTANTANEOUS VELOCITY. In Section 2.4 we will derive differentiation formulas that can be used to show that the instantaneous velocity of the plane in Example 6 is, in fact, exactly 57 miles per hour at $t = 3$.

Interactive Examples 2.2

Interactive solutions are on the web page http//www.math.ucsd.edu/~ashenk/.[†]

1. **(a)** Find the average rate of change of $y = \ln x$ for $0.5 \le x \le 4.5$. **(b)** Draw the corresponding secant line with the graph of the function.

2. The downward velocity of a parachutist is $v = 80 - 80(4^{-t/2})$ feet per second t seconds after she jumps from an airplane. **(a)** What is the average rate of change of her downward velocity with respect to t for $1 \le t \le 4$? **(b)** Give an equation of the corresponding secant line. **C** **(c)** Generate the secant line with the graph of the velocity in the window $0 \le t \le 7, 0 \le v \le 100$. Copy them on your paper.

3. The table below gives the average selling price of condominiums in downtown San Diego at the beginning of years 2001 through 2006.[(3)] What was the average rate of change of the price with respect to time **(a)** from the beginning of 2001 to the beginning of 2004 and **(b)** from the beginning of 2004 to the beginning of 2006?

Year	2001	2002	2003	2004	2005	2006
Price	$388,000	$416,000	$465,000	$610,000	$454,000	$490,000

[†] In the published text the interactive solutions of these examples will be on an accompanying CD disk which can be run by any computer browser without using an internet connection.

[(3)] Data adapted from *The San Diego Union*, May 7, 2006.

4. Figure 13 shows the graph of the annual United States trade deficit from 1980 to 2005.[4] What was the approximate average rate of change of the trade deficit from 1980 to 2005?

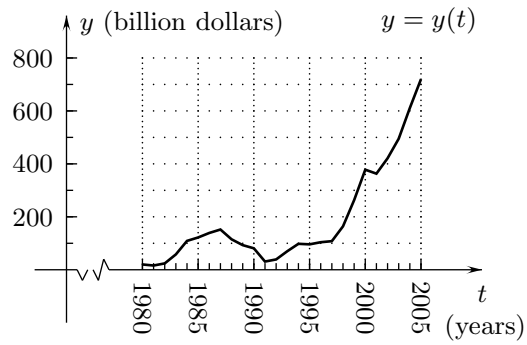


FIGURE 13

Exercises 2.2

[A]Answer provided. [O]Outline of solution provided. [C]Graphing calculator or computer required.

CONCEPTS:

Give units with answers when appropriate in the following exercises.

1. Why is the average rate of change of $y = f(x)$ with respect to x from a to b not defined when $a = b$?

2. Give an algebraic explanation of why the average rate of change of $y = f(x)$ with respect to x from b to a for $a \neq b$ is the same as the average rate of change of $y = f(x)$ with respect to x from a to b.

3. What can you say about a function $y = g(x)$ if its rate of change with respect to x from a to b is zero for all nonequal numbers a and b?

4. Find a formula for the function $y = h(x)$ such that $h(0) = 5$ and its average rate of change with respect to x from $x = 0$ to $x = b$ is 1 for all $b \neq 0$.

BASICS:

5.[O] An object is at $s = 50 - 50/t$ (yards) on an s-axis at time t (minutes) for $t \geq 1$. Find its average velocity for $1 \leq t \leq 5$. Sketch the graph of $s = s(t)$ without generating it on a calculator or computer and draw the secant line whose slope is the average velocity. (Plot the points on the curve at $t = 1, 2, 3, 4, 5$.)

6.[O] Estimate the velocity at $t = 2$ of the object in Exercise 5 by calculating its average velocity for $1.99 \leq t \leq 2$, for $2 \leq t \leq 2.01$, and for $1.99 \leq t \leq 2.01$.

7.[O] Values of a function $y = Q(x)$ are given in the table below. Find the average rate of change of Q with respect to x **(a)** for $0 \leq x \leq 0.2$, **(b)** for $0.2 \leq x \leq 0.4$, and **(c)** for $0 \leq x \leq 0.4$.

x	0	0.1	0.2	0.3	0.4
$Q(x)$	50.3	52.4	55.0	49.3	50.3

[4]Data from the U.S. Census Bureau, Foreign Trade Division, 2006.

8.[O] The graph of a function W is shown in Figure 14. Find the average rates of change of W with respect to x **(a)** for $2 \le x \le 5$, **(b)** for $1 \le x \le 3$, and **(c)** for $2.50 \le x \le 2.51$.

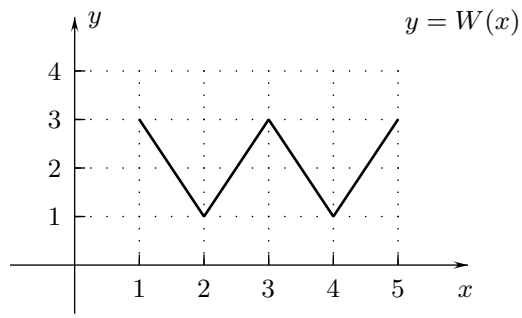


FIGURE 14

In Exercises 9 through 12 **(a)** find the average rate of change of the function in the given interval and **(b)** draw the corresponding secant line with the graph of the function.

9.[O] $y = x^6$ for $-1 \le x \le 1$

10.[A] $y = \dfrac{1}{x^2+1}$ for $-1 \le x \le 1$

11. $y = \sin x$ for $0 \le x \le \frac{3}{2}\pi$

12. $y = e^x$ for $1 \le x \le 3$

13.[O] A motorcyclist, riding away from his home town, is $s(t) = 10t^2$ (miles) from the city limits at time t (hours) for $0 \le t \le 3$. **(a)** What is his average velocity away from the town for $0 \le t \le 3$? **(b)** Draw the graph of $s = s(t)$ and the secant line whose slope is the average velocity from part (a). **(c)** Does he speed up or slow down during the ride? **(d)** What constant velocity would give him the same average velocity for $0 \le t \le 3$?

14.[O] A ball rolling toward the edge of a table is $s(t) = 10/t$ centimeters from the edge at time t (seconds) for $t \ge 1$. **(a)** What is its average velocity away from the edge for $2 \le t \le 10$? **(b)** Give an equation of the corresponding secant line. [C]**(c)** Generate the graph and the secant line together in the window $0 \le t \le 15, 0 \le s \le 12$ and copy them on your paper.

15. A \$1000 deposit that earns 5% annual interest compounded continuously grows to $B = 1000e^{0.05t}$ dollars in t years. **(a)** What is the average rate of change of the balance B with respect to time for $0 \le t \le 40$? **(b)** Give an equation of the corresponding secant line. [C]**(c)** Generate the graph and the secant line together in the window $0 \le t \le 50, \le B \le 10{,}000$ and copy them on your paper

16.[A] The following table gives the earth's distance from the sun at three month intervals during one year.[5] The distances are not all the same because the earth's orbit is not a circle. It is an ellipse with the sun at one focus. What is the average rate of change with respect to time (months) of the distance (kilometers) from the earth to the sun **(a)** between January 1 and July 1 and **(b)** between July 1 and the following January 1?

Date	January 1	April 1	July 1	October 1	January 1
Distance ($\times 10^8$ km)	1.4710	1.4949	1.5208	1.4977	1.4710

[5] *Space Mathematics*, Washington, D.C.: National Aeronautics and Space Administration, 1985, p 20.

17.[A] The average number of words in a human's vocabulary at several ages are listed below.[6] Which is greater, the average rate of change with respect to age of the size of vocabulary between ages 5 and 6 or between ages 6 and 18?

Age	1	2	3	4	5	6	18
Words in vocabulary	3	300	900	1,500	2,000	2,500	15,000

18.[O] Figure 15 is the graph of a cow's weight as a function of her age t.[7] Find the approximate average rate of change of her weight with respect to time from age two to the time when she weighed the most. Then draw the corresponding secant line on the graph.

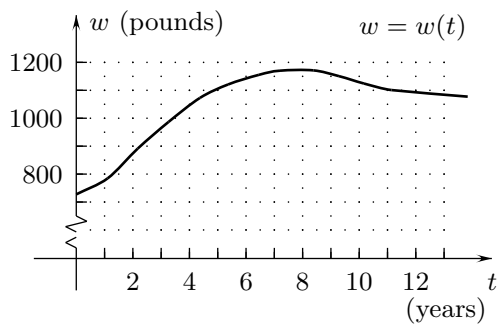


FIGURE 15

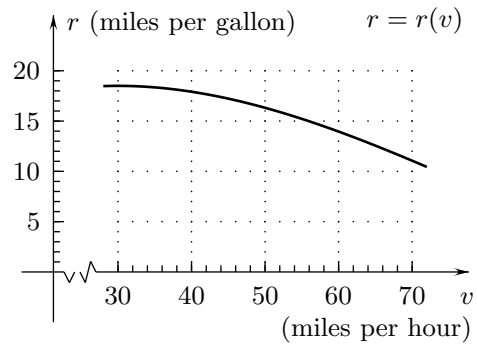


FIGURE 16

19.[A] Figure 16 shows the graph of the gasoline mileage $r = r(v)$ (miles per gallon) of a car as a function of its velocity v (miles per hour).[8] What is the approximate average rate of change of the mileage with respect to velocity for $40 \le v \le 60$? Draw the corresponding secant line with the graph of $r = r(v)$.

[C] **20.**[A] A tank contains $V(t) = 24t - 3t^2$ gallons of water at time t (hours) for $t \ge 0$. **(a)** Generate the graph of $V = V(t)$ in the window $0 \le t \le 6, 0 \le V \le 55$ with V-scale $= 10$. Use a graphing calculator or computer to find the time T when the volume in the tank is a maximum. (T is an integer.) **(b)** What is the maximum volume? **(c)** Based on the shape of the graph, is water flowing into the tank more rapidly at $t = 1$ or at $t = 3$? Check your answer by estimating the rate of flow at $t = 1$ or at $t = 3$ with average rates of change of the volume with respect to t over short time intervals including $t = 1$ and $t = 3$.

[C] **21.** The temperature in a room is $T(t) = 85 - 50t^3(3^{-t})$ degrees Fahrenheit at time t (minutes). **(a)** Generate the graph of $T = T(t)$ for $0 \le t \le 7$, $0 \le T \le 100$ with T-scale $= 10$. Use a <minimum> or <trace> command or the graph to find the approximate time t when the temperature is a minimum. **(b)** Describe how the temperature varies from $t = 0$ to $t = 7$. **(c)** Based on the graph, at approximately what time is the temperature falling the most rapidly? Use an average rate of change to estimate how fast it is falling at that time.

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[6] *Great Psychologists*, S. S. Sargent, New York: Barnes & Noble, 1958, p 90.

[7] Data adapted from *The Science of Animals that Serve Mankind* by J.R. Campbell and J.F. Lasley, McGraw-Hill Book Company, N.Y., 1975, p. 207. Source: Missouri Agricultural Experimental Station.

[8] Data adapted from *The effect of speed on gasoline consumption rates*, Washington, DC: U. S. Department of Transportation, Federal Highway Administration, October, 1973, p. 6.

EXPLORATION:

22.[A] The deep space probe Pioneer 10 took 21 months to travel the 994 million kilometers from Mars to Jupiter.[9] **(a)** What was its average velocity, measured in millions of kilometers per month? **(b)** What was its average velocity, measured in kilometers per hour?

23.[O] A total of 4,718 million pounds of fish were caught by United States fishermen in 1950, of which 1,001 million pounds were caught by fishermen from New England and 492 million pounds by fishermen from Alaska. In 1993, 10,436 million pounds were caught, of which 605 million pounds were caught from New England and 5,906 million pounds were caught from Alaska.[10] What were **(a)** the percent change from 1950 to 1993 of the total annual catch by U. S. fishermen, **(b)** the average rate of change with respect to time from 1950 to 1993 of the annual catch by fishermen from New England, and **(c)** the fraction of the U. S. catch that was made by Alaska fishermen in 1993?

24.[A] In 1988 there where 26,721 applicants to medical schools in the United States and 15,569 applicants were admitted. There was a 70% increase in the number of applicants and a 2% increase in the number of admissions from 1988 to 1994.[11] What were the average rates of change with respect to time of the numbers of applicants and admissions from 1988 to 1994?

25. The graph drawn with a heavy line in Figure 17 gives the price of a pound of coffee in New York City with the values interpolated from data every five years from 1900 through 1985. The graph drawn with a finer line gives the price of a dozen eggs.[12] What was the approximate average rate of change with respect to time from 1900 to 1985 of the total cost of three pounds of coffee and two dozen eggs?

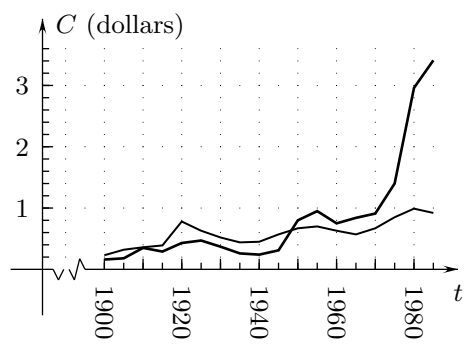

FIGURE 17

26.[A] The radius r and area $A = \pi r^2$ of an expanding circle are given as functions of the time t in the table below. What are **(a)** the average rate of change of r with respect to t for $20 \le t \le 60$, **(b)** the average rate of change of A with respect to r for $5 \le r \le 9$, and **(c)** the average rate of change of A with respect to t for $20 \le t \le 60$? **(d)** How are the answers to parts (a), (b) and (c) related? Give units and exact answers, using π as necessary.

t (seconds)	0	20	40	60
r (feet)	2	5	8	9
A (square feet)	4π	25π	64π	81π

[9] *Space Mathematics*, Washington, D.C.: National Aeronautics and Space Administration, 1985, p 21.
[10] *Los Angeles Times*, March 29, 1995. Source: National Marine Fisheries Service.
[11] *Los Angeles Times*, June 4, 1995. Source: Assn. of American Medical Colleges.
[12] Data from *The Value of a Dollar*, Detroit: Gale Research Inc., 1994, pp. 65–546.

27 An airplane is 3000 feet above the ground three minutes after taking off from sea level. The air pressure is 14.7 pounds per square inch at sea level and 12.9 pounds per square inch at 3000 feet.[13] What are **(a)** the average rate of change of the plane's altitude with respect to time during the first three minutes of the flight? **(b)** the average rate of change of the air pressure with respect to altitude from sea level to 3000 feet, and **(c)** the average rate of change with respect to time of the air pressure on the plane during the first three minutes of flight? **(d)** How are the answers to parts (a), (b), and (c) related? (Give the units.)

28. A professional baseball pitcher throws a baseball straight up in the air with a velocity of 80 feet per second, the ball comes straight down, and he catches it at the point from which he released it. With the simplifying assumption that there is no air resistance so that the only force on the ball is gravity, then the ball is $s(t) = 80t - 16t^2$ feet above the pitcher's hand t seconds after he throws it, up to time $t = 5$, when he catches it. The graph of $s = s(t)$ is the upper curve in Figure 18. **(a)** Find in this case the approximate velocity of the ball when he catches it by finding its average upward velocity for $4.999 \le t \le 5$. **(b)** If we include air resistance, then the ball's height above the pitcher's hand at time t is $s_R(t) = 1200 - 1200e^{-0.2t} - 160t$ feet, where the subscript "R" is used to suggest "resistance." The graph of this function is the lower curve in Figure 17.† Estimate with this model the ball's upward velocity at $t = 4.37$, just before he catches it, by finding its upward velocity for $4.37 \le t \le 4.371$. **(c)** Decribe how air resistance affects the flight of the ball.

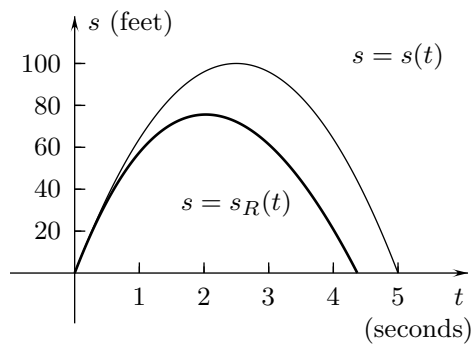

FIGURE 18

C **29.** A helium balloon is released at time $t = 0$ (seconds). It rises at first but then descends because the helium leaks slowly from it. It is $h = 20 + 10t - t^2$ feet above the ground at time t until it it reaches the ground. **(a)** Use the Quadratic Formula to find the exact and approximate decimal value of the time when it reaches the ground. **(b)** Generate the graph of $h(t)$ in the window $0 \le t \le 12, -5 \le h \le 50$ with h-scale $= 10$ and find the time when the balloon is at its highest point (an integer value of t). **(c)** How high does the balloon go? **(d)** Estimate the balloon's upward velocity at $t = 0$, at $t = 5$, and when it reaches the ground by calculating its average velocity in short time intervals.

C **30.** A fly is $h(t) = 2t - \frac{1}{15}t^2 + 3\sin t$ feet above a picnic table at time t (seconds) **(a)** Generate the graph of $h(t)$ in the window $0 \le t \le 25$, $-2 \le h \le 25$ with the line $h = 10$ and use an <intersect> or <trace> operation or the graph to find the approximate times a and b when the fly is 10 feet above the table. **(b)** What is the average rate of change with respect to time of the fly's height above the table from time a to time b? **(c)** Estimate the fly's upward velocity at $t = a$ and $t = b$ by calculating average rates of change of $h(t)$ with respect to t in short time intervals.

(End of Section 2.2)

[13] *Encyclopædia Britannica*, Vol. I, 15th Edition, Chicago: Encyclopædia Britannica, 1974, p. 627.

† This problem uses a mathematical model in which air resistance is assumed to be proportional to the ball's velocity.