Section 2.3

(3/19/08)

Tangent lines, rates of change, and derivatives

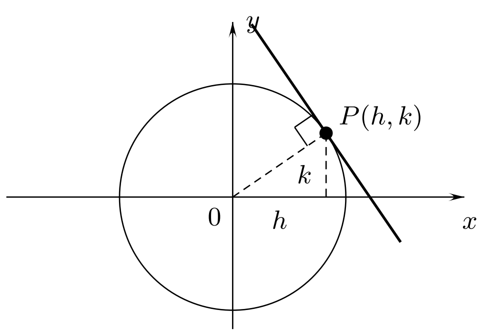
OVERVIEW: *The derivative was developed in the seventeenth century for determining tangent lines to curves and the velocity of moving objects in cases that could not be handled with geometry and algebra alone. The ancient Greeks had used reasoning similar to that in modern plane geometry to study tangent lines to circles and other special curves. This approach could not be used, however, to find tangent lines to most curves. We will see in this section how tangent lines can be found as the limiting positions of secant lines and how instantaneous velocity and other instantaneous rates of change can be found as limits of average velocities and average rates of change. This leads to the definition of the derivative as the slope of a tangent line and as an instantaneous rate of change.*

Topics:

- ***Tangent lines, derivatives, and instantaneous rates of change***
- ***Predicting a derivative by calculating difference quotients***
- ***Finding exact derivatives***
- ***Equations of tangent lines***
- ***The ΔxΔf-formulation of the definition***
- ***Finding a rate of change from a tangent line***
- ***A secant-line program***

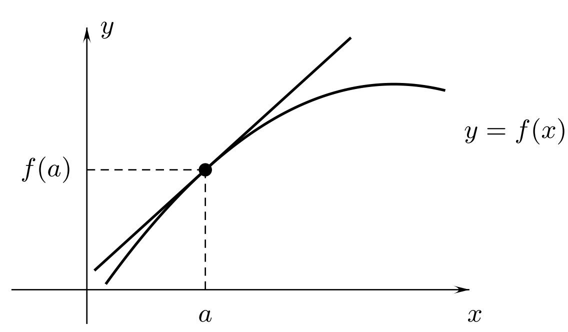
Tangent lines, derivatives, and instantaneous rates of change

Euclid (c. 300 BC) defined a tangent line to a circle at a point P to be the line that intersects the circle at only that point (Figure 1).† The tangent line is outside the circle except at the point of tangency. This implies that the tangent line is perpendicular to the radius at P because P is the closest point on the tangent line to the center O of the circle and consequently is at the foot of the perpendicular line from O to the tangent line. This property enables us to find the slope of the tangent line: if, for example, the center of the circle is the origin in an xy-plane, as in Figure 1, and P has coordinates (h, k), then the slope of the radius OP is k/h, and consequently the slope of the perpendicular tangent line is $-h/k$.



Tangent line at P

FIGURE 1



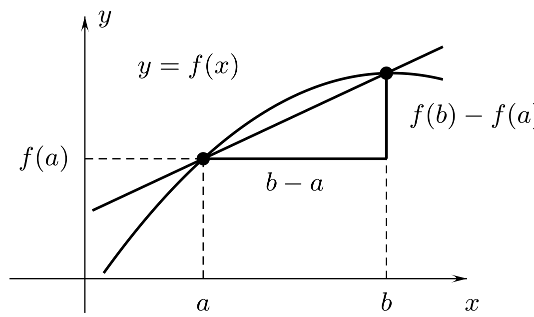
Tangent line at $x = a$

FIGURE 2

†The use of the term "tangent" in "tangent line" comes from the Latin *tangere*, to touch.

The graph $y = f(x)$ in Figure 2 does not have a similar geometric property that could be used to find its tangent lines. Instead, we find a tangent line as a limit of secant lines. As we saw in the last section, the slope of the secant line through the points at $x = a$ and $x = b$ for $b \neq a$ (Figure 3) equals the rise $f(b) - f(a)$ from the point at $x = a$ to the point at $x = b$, divided by the corresponding run $b - a$:

$$[\text{Slope of the secant line}] = \frac{f(b) - f(a)}{b - a}. \tag{1}$$

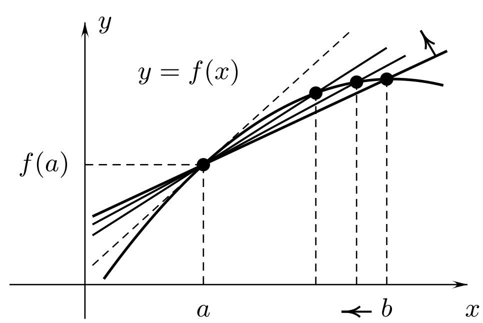


[Slope of the secant line]

$= \dfrac{f(b) - f(a)}{b - a}.$

FIGURE 3

Imagine what happens as the number b approaches a (Figure 4). The secant line turns about the fixed point at $x = a$. If its slope **(1)** has a finite limit, then the secant line approaches the line with that slope—the dashed diagonal line in Figure 4—and this line is the tangent line to the graph of f at $x = a$ that is shown in Figure 2.



Secant lines approaching

the tangent line as $b \to a$

FIGURE 4

The limit as b tends to a of the slope of the secant line is called the DERIVATIVE of f at a and is denoted $f'(a)$:

$$f'(a) = \lim_{b \to a} \frac{f(b) - f(a)}{b - a}.$$

Consequently, the tangent line at $x = a$ to the graph of f is the line through the point at $x = a$ that has slope equal to the derivative $f'(a)$.

How about rates of change? We also saw in the last section that the slope **(1)** of the secant line is the average rate of change of f with respect to x from $x = a$ to $x = b$. If b is very close to a, then this average rate of change is a good measure of how rapidly the function is changing at a. We define its limit $f'(a)$ as b tends to a to be the INSTANTANEOUS RATE OF CHANGE of the function with respect to x at $x = a$.

These definitions can be summarized as follows:

Definition 1 *The derivative of $y = f(x)$ at a is*

$$f'(a) = \lim_{b \to a} \frac{f(b) - f(a)}{b - a} \tag{2}$$

provided this limit exists and is finite. The derivative $f'(a)$, if it exists, is the slope of the tangent line to the graph of f at $x = a$ and is the instantaneous rate of change of f with respect to x at a.

If x and $f(x)$ have units, then the units used for the derivative are those used for $f(x)$ divided by those used for x.

The derivative is usually referred to as the “rate of change” rather than as the “instantaneous rate of change” as in Definition 1 except when it is being discussed with average rates of change.

The process of finding derivatives is called DIFFERENTIATION.

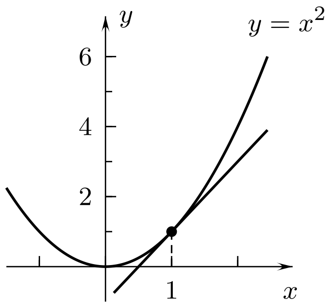
The average rate of change **(1)** is often called a DIFFERENCE QUOTIENT because it is a quotient of the differences $f(b) - f(a)$ and $b - a$. We then have three terms for average rate of change and three terms for (instantaneous) rate of change, which are related as shown the following diagram, where the vertical arrows indicate that b is approaching a:

$\dfrac{f(b) - f(a)}{b - a}$	=	Average rate of change	=	Slope of the secant line	=	The difference quotient
$\downarrow$		$\downarrow$		$\downarrow$		$\downarrow$
$f'(a)$	=	Instantaneous rate of change	=	Slope of the tangent line	=	The derivative

INSTANTANEOUS VELOCITY is the instantaneous rate of change with respect to time of an object’s coordinate $s = s(t)$ on an s-axis and is the limit of average velocities.

Predicting a derivative by calculating difference quotients

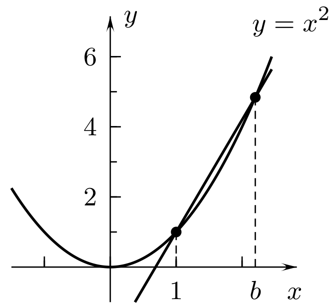
Figure 5 shows the graph of $f(x) = x^2$ and its tangent line at $x = 1$, whose slope equals the derivative $f'(1)$. In the next example we predict the value of this derivative by calculating the slope of the secant line (Figure 6) for values of b close to 1.



Tangent line

Slope $= f'(1)$

FIGURE 5



Secant line

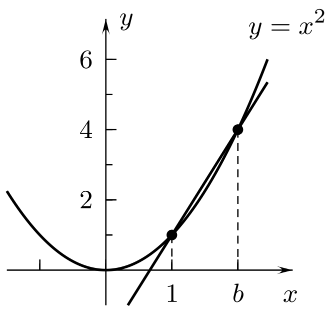
Slope $= \dfrac{f(b) - f(1)}{b - 1}$

FIGURE 6

Example 1 Predict the derivative $f'(1)$ for $f(x) = x^2$ by calculating the average rate of change $\dfrac{f(b) - f(1)}{b - 1} = \dfrac{b^2 - 1}{b - 1}$ for $b = 2, b = 1.5, b = 1, b = 0.1, b = 0.001$, and $b = 0.00001$.

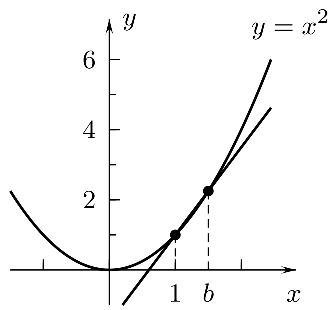
SOLUTION The average rates of change in the table below suggest that the derivative $f'(1) = \lim_{b \to 1} \dfrac{b^2 - 1}{b - 1}$ is 2. The secant lines corresponding to the values in the table are shown in Figures 7 through 12. The secant lines for $b = 0.001$ and $b = 0.00001$ in Figures 11 and 12 are indistinguishable and look like the tangent line in Figure 5 because these values of b are very close to 1. □

b	2	1.5	1.25	1.1	1.001	1.00001
$\dfrac{b^2 - 1}{b - 1} =$	3	2.5	2.25	2.1	2.001	2.00001



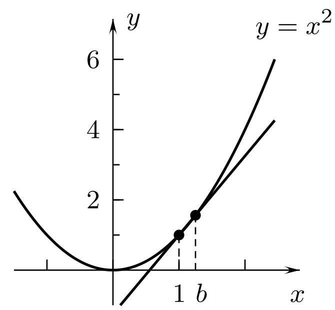
$b = 2$

FIGURE 7



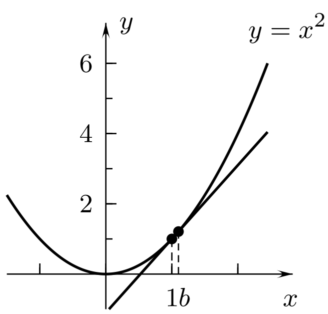
$b = 1.5$

FIGURE 8



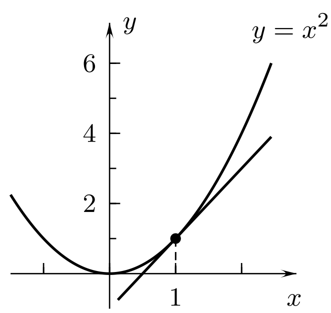
$b = 1.25$

FIGURE 9



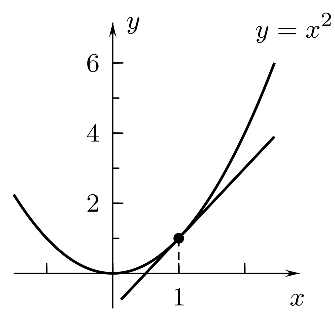
$b = 1.1$

FIGURE 10



$b = 1.001$

FIGURE 11



$b = 1.00001$

FIGURE 12

Finding exact derivatives

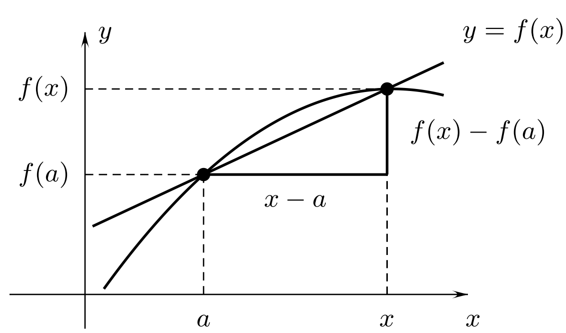
To make the algebra in finding exact derivatives easier to follow, we use x instead of b for the x-coordinate of the variable point in definition **(2)**. Then the secant line passes through $(a, f(a))$ and $(x, f(x))$, as shown in Figure 13, and definition **(2)** takes the form

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}. \tag{3}$$

Secant line of slope

$$\frac{f(x) - f(a)}{x - a}$$

FIGURE 13



Example 2 Use definition **(3)** to find the derivative of $f(x) = x^2$ at $x = 1$.

SOLUTION Since $f(1) = 1^2 = 1$, definition **(3)** with $a = 1$ gives

$$f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1}. \qquad \textbf{(4)}$$

We cannot find this limit by taking the limit of the numerator and denominator because they both tend to zero and $0/0$ has no meaning. Instead, as we saw with similar examples in Section 1.2, we can factor the numerator to obtain

$$\frac{x^2 - 1}{x - 1} = \frac{(x + 1)(x - 1)}{x - 1} \text{ for } x \neq 1.$$

Canceling one factor of the numerator with the denominator then gives

$$\frac{x^2 - 1}{x - 1} = x + 1 \text{ for } x \neq 1.$$

Because the polynomial $y = x + 1$ is continuous at $x = 1$, its limit as $x \to 1$ is equal to its value at $x = 1$. The last equation and **(4)** yield

$$f'(1) = \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} (x + 1) = \Big[x + 1\Big]_{x=1} = 2.$$

The derivative of $f(x) = x^2$ at $x = 1$ is 2, as we predicted in Example 1. □

Example 3 Use definition **(3)** to find $g'(0)$ for $g(x) = 3x - x^3$.

SOLUTION Since $g(0) = 3(0) - 0^3 = 0$, definition **(3)** with g in place of f and $a = 0$ yields

$$g'(0) = \lim_{x \to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \to 0} \frac{(3x - x^3) - 0}{x} = \lim_{x \to 0} \frac{3x - x^3}{x}.$$

To express the last quotient as a function whose limit we can find, we factor x from the numerator and cancel it with the denominator to have

$$\frac{3x - x^3}{x} = \frac{x(3 - x^2)}{x} = 3 - x \text{ for } x \neq 0.$$

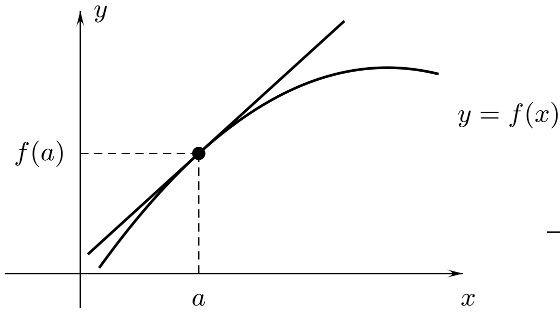
Since the polynomial $y = 3 - x$ is continuous at $x = 0$, its limit as $x \to 0$ is its value at $x = 0$, and we obtain

$$g'(0) = \lim_{x \to 0} \frac{3x - x^3}{x} = \lim_{x \to 0} (3 - x^2) = \Big[3 - x^2\Big]_{x=0} = 3 - 0^2 = 3. \ \square$$

Equations of tangent lines

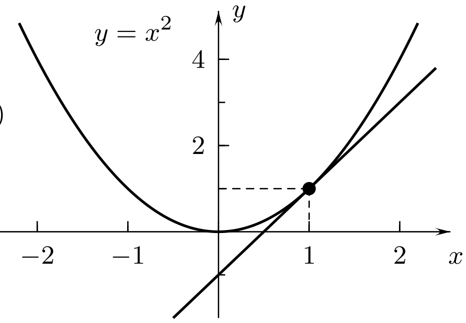
The tangent line to $y = f(x)$ at $x = a$ in Figure 14 passes through the point $(a, f(a))$ and has slope $f'(a)$. By the point-slope formula, it has the equation,

$$y = f(a) + f'(a)(x - a). \tag{5}$$



$y = f(a) + f'(a)(x - a)$

FIGURE 14



Tangent line at $x = 1$

FIGURE 15

Example 4 Give an equation of the tangent line to the graph of $f(x) = x^2$ at $x = 1$ and draw it with the graph of the function.

SOLUTION The value $f'(1) = 2$ of the derivative was found in Example 2. Also $f(1) = 1^2 = 1$, so by **(5)** with $a = 1$, the tangent line has the equation $y = 1 + 2(x - 1)$, which can be rewritten as $y = 2x - 1$. The curve and tangent line are shown in Figure 15. □

Example 5 Give an equation of the tangent line to the graph of $H(x) = 1/x$ at $x = 1$ and draw the curve with the tangent line.

SOLUTION We need to rewrite the difference quotient

$$\frac{H(x) - H(1)}{x - 1} = \frac{\frac{1}{x} - 1}{x - 1} \quad \text{for } x \neq 0, 1 \tag{6}$$

so we can find its limit $H'(1)$ as $x \to 1$. We first simplify the fractions to obtain for $x \neq 0, 1$,

$$\begin{aligned} \frac{H(x) - H(1)}{x - 1} &= \frac{\frac{1}{x} - 1}{x - 1} = \frac{1}{x - 1}\left(\frac{1}{x} - 1\right) \\ &= \frac{1}{x - 1}\left(\frac{1 - x}{x}\right) = \frac{-(x - 1)}{x(x - 1)}. \end{aligned}$$

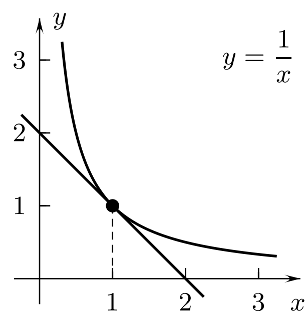
Next, we cancel the factors $x - 1$ in the numerator and denominator to have for $x \neq 0, 1$,

$$\frac{H(x) - H(1)}{x - 1} = \frac{-1}{x}.$$

Then, because the rational function $y = -1/x$ is continuous at $x = 1$, its limit as $x \to 1$ is its value at $x = 1$ and

$$H'(1) = \lim_{x\to 1} \frac{H(x) - H(1)}{x - 1} = \lim_{x\to 1} \frac{-1}{x}$$
$$= \left[\frac{-1}{x}\right]_{x=1} = -1.$$

The tangent line has the equation $y = H(1) + H'(1)(x - 1)$ and since $H(1) = 1/1 = 1$, the tangent line is $y = 1 - (x - 1)$. We rewrite this equation as $y = 2 - x$ to see that it is the line of slope -1 with y-intercept 2. The curve and tangent line are drawn in Figure 16. □



The tangent line $y = 2 - x$

FIGURE 16

The next example involves a function with parameters (constants) in its formula.

Example 6 What is the derivative of $J(x) = \dfrac{ax}{x^2 + b}$ at $x = 0$ if a and b are nonzero constants?

SOLUTION By Definition 1,

$$J'(0) = \lim_{x\to 0} \frac{J(x) - J(0)}{x - 0}. \tag{7}$$

Because $J(0)$ is zero, the difference quotient on the right of **(7)** can be written

$$\frac{J(x) - J(0)}{x - 0} = \frac{1}{x}[J(x)] = \frac{1}{x}\left[\frac{ax}{x^2 + b}\right] \quad \text{for } x \neq 0.$$

Cancelling the x's in the numerator and denominator then gives

$$\frac{J(x) - J(0)}{x - 0} = \frac{a}{x^2 + b} \quad \text{for } x \neq 0.$$

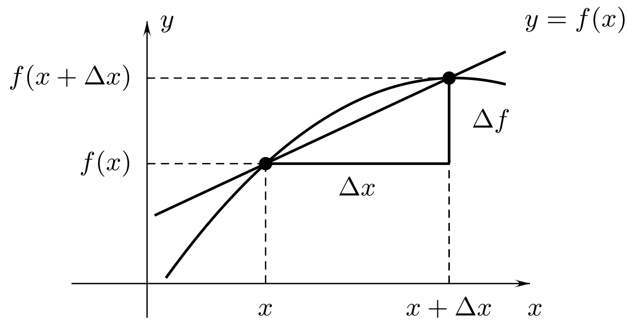
Since b is not zero, the rational function $y = \dfrac{a}{x^2 + b}$ is defined and continuous at $x = 0$ and definition **(7)** yields

$$J'(0) = \lim_{x\to 0} \frac{a}{x^2 + b} = \left[\frac{a}{x^2 + b}\right]_{x=0} = \frac{a}{0^2 + b} = \frac{a}{b}. \ \square$$

The Δx*-definition of the derivative*

It is often convenient to use definition **(2)** of the derivative with different notation for the rise and run on the secant line. We let x be the fixed value where we want to find the derivative and denote the x-coordinate of the variable point by $x+\Delta x$, where Δ is the capital Greek letter delta. Then Δx is the run from $(x, f(x))$ to $(x+\Delta x, f(x+\Delta x))$ on the secant line, as shown in Figure 17. We denote the corresponding rise $f(x+\Delta x)-f(x)$ by Δf.† With this notation, the definition of the derivative becomes

$$f'(x) = \lim_{\Delta x\to 0}\frac{\Delta f}{\Delta x} = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}. \tag{8}$$



Secant line of slope

$$\frac{\Delta f}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

FIGURE 17

Formulation **(8)** of the definition is especially convenient for finding a derivative at a variable point x.

Example 7 Use formulation **(8)** of the definition to find the derivative of $f(x)=1/x$ at an arbitrary $x\neq 0$.

SOLUTION We fix $x\neq 0$ and set $f(x)=1/x$ in **(8)** to have

$$f'(x) = \lim_{\Delta x\to 0}\frac{\dfrac{1}{x+\Delta x}-\dfrac{1}{x}}{\Delta x}. \tag{9}$$

To convert the three-layer fraction on the right of **(9)** into a product of two-level fractions, we replace division by Δx with multiplication by $1/\Delta x$. Then we take a common denominator and simplify the result:

$$\begin{aligned}\frac{\dfrac{1}{x+\Delta x}-\dfrac{1}{x}}{\Delta x} &= \frac{1}{\Delta x}\left[\frac{1}{x+\Delta x}-\frac{1}{x}\right] = \frac{1}{\Delta x}\left[\frac{x-(x+\Delta x)}{x(x+\Delta x)}\right]\\ &= \frac{1}{\Delta x}\left[\frac{-\Delta x}{x(x+\Delta x)}\right] \quad \text{for } \Delta x\neq 0.\end{aligned}$$

Next, we cancel the Δx's in the numerator and denominator to have

$$\frac{\dfrac{1}{x+\Delta x}-\dfrac{1}{x}}{\Delta x} = \frac{-1}{x(x+\Delta x)} \quad \text{for } \Delta x\neq 0. \tag{10}$$

†Notice that each of Δx and Δf denotes a single number and not a product of a Δ and x or f. Think of Δ as "the change in," so that Δx is the change in x and and Δf is the change in f.

Since x is not zero, the rational function of Δx on the right of **(10)** is continuous at $\Delta x = 0$, so that by **(10)**,

$$f'(x) = \lim_{\Delta x \to 0} \frac{-1}{x(x+\Delta x)} = \left[\frac{-1}{x(x+\Delta x)}\right]_{\Delta x = 0} = -\frac{1}{x^2}. \square.$$

Finding a rate of change from a tangent line

The function $A = A(t)$ whose graph is shown in Figure 18 gives the percentage of alcohol in a person's blood t hours after he has consumed three fluid ounces of alcohol.[1] As you can see from the graph, the blood-alcohol level rises from 0% to about 0.22% in about two hours, and then drops close to 0.01% after ten more hours.

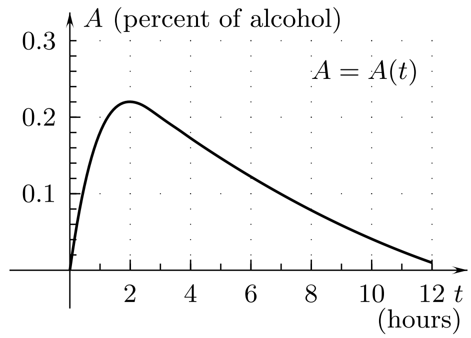


FIGURE 18

Example 8 The tangent line at $t = 1$ in Figure 19 passes through the points $P = (1, 0.18)$ and $Q = (2, 0.28)$. What is the (instantaneous) rate of change of the person's blood alcohol level with respect to time one hour after consuming the alcohol?

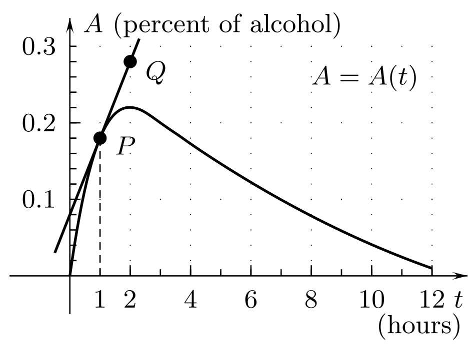


FIGURE 19

Solution The rate of change $A'(1)$ of the blood-alcohol level after one hour is the slope of the tangent line in Figure 19. We can find its exact value because we are given the exact coordinates of two points on the line. We obtain

$$A'(1) = \frac{(0.28 - 0.18)\ \text{percent}}{(2-1)\ \text{hour}} = 0.1 \text{ percent per hour.} \square$$

The next example uses the technique of rationalization of differences of square roots that was used to find limits in Section 1.2.

[1] Data adapted from *Encyclopædia Britannica*, Vol. 1, Chicago: Encyclopædia Britannica, Inc., 1965, p. 548.

Example 9 Find the derivative of $y = \sqrt{x}$ at $x = 5$.

SOLUTION By definition **(3)** with $a = 5$ and $y = \sqrt{x}$ in place of $y = f(x)$,

$$y'(5) = \lim_{x \to 5} \frac{y(x) - y(5)}{x - 5} = \lim_{x \to 5} \frac{\sqrt{x} - \sqrt{5}}{x - 5}.$$

In order to cancel the denominator, we need to first rationalize the difference $\sqrt{x} - \sqrt{5}$ of square roots in the numerator. We multiply and divide by the sum $\sqrt{x} + \sqrt{5}$ of the square roots to obtain for nonnegative $x \neq 5$,

$$\frac{y(x) - y(5)}{x - 5} = \frac{\sqrt{x} - \sqrt{5}}{x - 5} = \frac{(\sqrt{x} - \sqrt{5})(\sqrt{x} + \sqrt{5})}{(x - 5)(\sqrt{x} + \sqrt{5})} = \frac{(\sqrt{x})^2 - (\sqrt{5})^2}{(x - 5)(\sqrt{x} + \sqrt{5})}$$
$$= \frac{x - 5}{(x - 5)(\sqrt{x} + \sqrt{5})} = \frac{1}{\sqrt{x} + \sqrt{5}}.$$

Then, because $y = 1/(\sqrt{x} + \sqrt{5})$ is continuous at $x = 5$, we obtain

$$y'(5) = \lim_{x \to 5} \frac{1}{\sqrt{x} + \sqrt{5}} = \left[\frac{1}{\sqrt{x} + \sqrt{5}}\right]_{x=5} = \frac{1}{2\sqrt{5}}. \square$$

C *A secant line program*

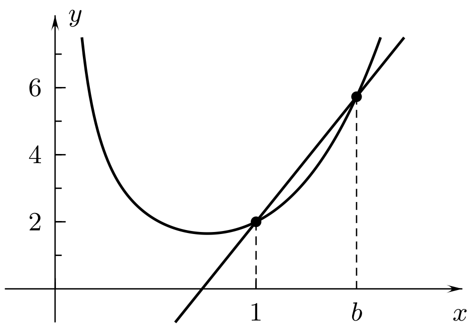
A secant line program for several types of calculators is available on the web page for this book.† The program illustrates the definition of the derivative by generating secant lines with graphs of the functions and calculating the difference quotients $\dfrac{f(b) - f(a)}{b - a}$ that are their slopes.

Example 10 Predict the derivative $f'(1)$ for $f(x) = x^4 + 1/x$ by using a secant-line program to calculate the difference quotient $\dfrac{f(b) - f(1)}{b - 1} = \dfrac{b^4 + 1/b - 2}{b - 1}$ for $b = 1.5, 1.2,$ $1.1, 1.001$, and $b = 1.00001$. Use the window $-0.25 \leq x \leq 2, -1 \leq y \leq 8$.

SOLUTION Enter the program and store the function $f(x) = x^4 + 1/x$ and define the window in your calculator, according to the instructions on the web page. Run the program with $a = 1$ and the specified values of b to obtain the secant lines in Figure 20 through 24 and their slopes in the following table. Because these slopes seem to be approaching 3 as b approaches 1, we predict that $f'(1) = 3$. (In the next section we will derive formulas that could be used to verify this prediction.) $\square$

x	1.5	1.2	1.1	1.001	1.00001
$\dfrac{f(b) - f(a)}{b - a} \doteq$	7.458333333	4.534666667	3.371909091	3.007003002	3.000070000

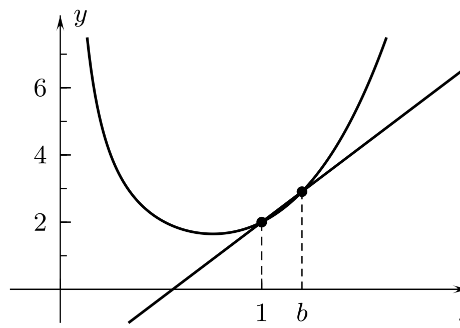
† www.math.ucsd.edu/∽ashenk/



$a = 1, b = 1.5$

[Slope] $\doteq$ 7.458333333

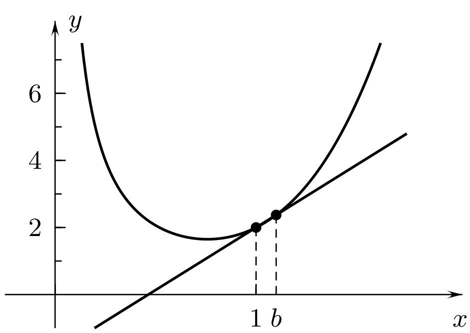
FIGURE 20



$a = 1, b = 1.2$

[Slope] $\doteq$ 4.534666667

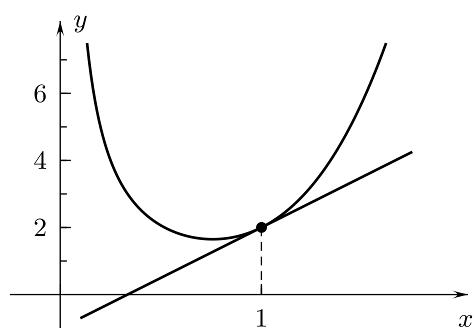
FIGURE 21



$a = 1, b = 1.1$

[Slope] $\doteq$ 3.731909091

FIGURE 22



$a = 1, b = 1.001$

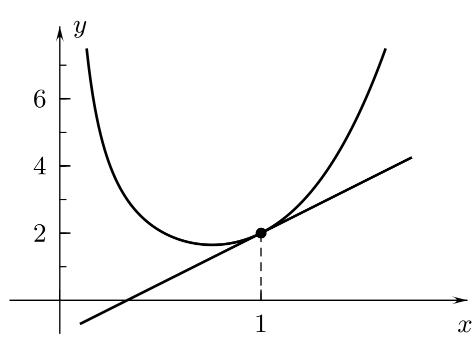
[Slope] $\doteq$ 3.007003002

FIGURE 23

$a = 1, b = 1.00001$

[Slope] $\doteq$ 3.000070000

FIGURE 24



Interactive Examples 2.3

Interactive solutions are on the web page http//www.math.ucsd.edu/˜ashenk/.[†]

1. A woman's trip is described by the mathematical model in which she is $s(t) = 60t + 3\sin(\pi t)$ miles east of a town at time t (hours) for $0 \le t \le 3$. Use an average velocity to estimate her instantaneous velocity toward the east at $t = 2$.

2. Predict the derivative $f'(1)$ for $f(x) = x^2 - 3x + 4$ by calculating the average rate of change of $f(x)$ from $x = 1$ to $x = b$ for $b = 1.5, 1.2, 1.1, 1.001$, and 1,00001. (If you use a secant line program on a calculator or computer, set the window to $-0.5 \le x \le 2.5, -2 \le y \le 4$.)

3. Use the definition $f'(a) = \lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ to find the derivative $f'(1)$ for $f(x) = x^2 - 3x + 4$..

4. Use results from the previous example to find an equation of the tangent line to the graph of $f(x) = x^2 - 3x + 4$ at $x = 1$. Then draw the curve and the tangent line..

5. Use the Δx-formulation of the definition to find a formula for $f'(x)$ with $f(x) = x^2 - 3x + 4$ as a function of x.

Exercises 2.3

[A]Answer provided. [O]Outline of solution provided. [C]Graphing calculator or computer required.

CONCEPTS:

1. Decribe how secant lines are related to instantaneous rates of change.

2. In some cases a tangent line at a point is the one line that intersects the curve only that point. Explain why this characterization of a tangent line does not apply to the tangent line at P in Figure 25.

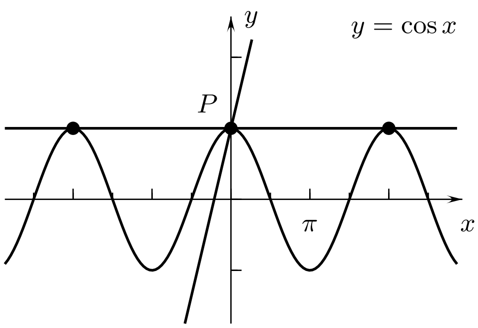


FIGURE 25

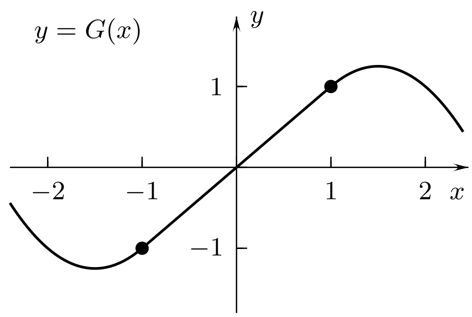


FIGURE 26

3. Figure 26 shows the graph of the function,

$$G(x) = \begin{cases} x^2 + 3x + 1 & \text{for} \quad x \le -1 \\ x & \text{for} \quad -1 < x < 1 \\ -x^2 + 3x - 1 & \text{for} \quad x \ge 1 \end{cases}$$

Explain why the tangent line to this curve at the origin cannot be characterized as the line that intersects the curve only at one point.

[†]In the published text the interactive solutions of these examples will be on an accompanying CD disk which can be run by any computer browser without using an internet connection.

4. Figure 27 shows the graph of blood-alcohol level from Example 9 and its tangent line at $t = 8$. The tangent line passes through the points $R = (4, 0.16)$ and $S = (8, 0.08)$. **(a)** What is the (instantaneous) rate of change of the person's blood alcohol level eight hours after consuming the alcohol? **(b)** Why is is negative?

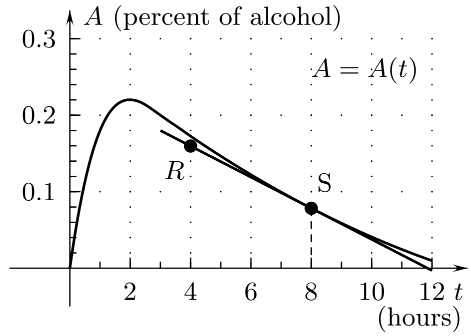


FIGURE 27

BASICS:

5.[O] Calculate the slope of the secant line through the points at $x = a$ and $x = b$ on the graph of $f(x) = 1/x^2$ with $a = 1$ and $b = 0.5, 0.75, 0.9, 0.999$ and 0.99999. Use the results to predict the value of the derivative $f'(1)$. If you use the secant-line program, set the window to $0 \leq x \leq 2, -1 \leq y \leq 5$.

6.[O] **(a)** Use formulation **(2)** of the definition to find the exact derivative of $f(x) = 1/x^2$ at $x = 1$. **(b)** Give an equation of the tangent line to $y = x^2$ at $x = 1$.

7.[O] Use the Δx-formulation **(8)** of the definition to find the derivative of $f(x) = 1/x^2$ at an arbitrary $x \neq 0$.

8.[O] Figure 28 shows the graph of a function $y = f(x)$ and two points on its tangent line at $x = 2$. What is the derivative $f'(2)$?

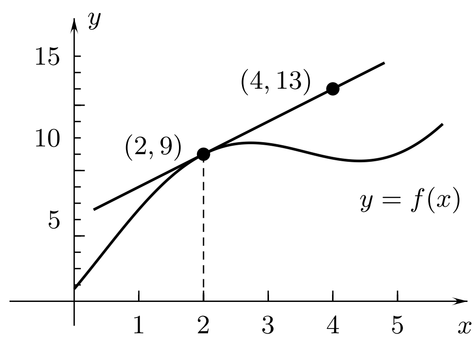


FIGURE 28

9.[O] Calculate the slope of the secant line through the points at $x = a$ and $x = b$ on the graph of $g(x) = 1/\sqrt{x}$ with $a = 1$ and $b = 0.1, 0.3, 0.9, 0.999$ and 0.99999. Use the results to predict the value of the derivative $g'(1)$. [C] If you use the secant-line program, set the window to $0 \leq x \leq 2, -1 \leq y \leq 4$.

10. Calculate the slope of the secant line through the points at $x = a$ and $x = b$ on the graph of $h(x) = x^3$ with $a = 1$ and $b = 0, 0.5, 0.9, 0.999$ and 0.99999. Use the results to predict the value of the derivative $h'(1)$. [C] If you use the secant-line program, set the window to $0 \leq x \leq 1.5, -1 \leq y \leq 2$.

In Exercises 11 through 14, calculate the difference quotient $\dfrac{f(b)-f(2)}{b-2}$ for $b = 0, 1, 1.9, 1.999$ and 1.99999, and use the results to predict the derivative $f'(2)$. **C** The given windows are to be used with the secant-line program.

11.[O] $f(x) = \dfrac{x^2+6}{x^2+1}$ $(-2 \le x \le 4, -1 \le y \le 8)$

12. $f(x) = (6x-4)^{1/3}$ $(-0.75 \le x \le 3, -2 \le y \le 5)$

13.[A] $f(x) = \dfrac{16}{\sqrt{x+2}}$ $(-1 \le x \le 3.5, -2 \le y \le 20)$

14. $f(x) = 10 + (x^2-4)^2$ $(-3.5 \le x \le 3.5, -5 \le y \le 50)$

Use the definition $f'(a) = \lim\limits_{x\to a} \dfrac{f(x)-f(a)}{x-a}$ to find the derivatives in Exercises 15 through 21.

15.[O] $g'(2)$ for $g(x) = x + 1/x$

16.[A] $G'(4)$ for $G(x) = 1/(5-2x)$

17.[O] $P'(0)$ for $P(x) = (x+4)/(x+2)$

18.[A] $y'(0)$ for $y(x) = x^{4/3}$

19. $g'(2)$ for $g(x) = 4$

20. $F'(1)$ for $F(x) = 6x + 3$

21. $f'(3)$ for $f(x) = -2x^2$

In Exercises 22 through 24 use the definition $f'(a) = \lim\limits_{x\to a} \dfrac{f(x)-f(a)}{x-a}$ to find the equation of the tangent line to the graph of the function at the given value of a. Then draw the graph of the function with the tangent line.

22.[O] $f(x) = 2/x^2$ at $a = -1$

23.[A] $f(x) = 4 - x^2$ at $a = -1$

24. $f(x) = -\frac{1}{4}x^3 - 2$ at $a = 0$

In Exercises 25 through 29 use the Δx-formulation of the definition to find the derivatives.

25.[O] $f'(3)$ for $f(x) = 1/(x+2)$

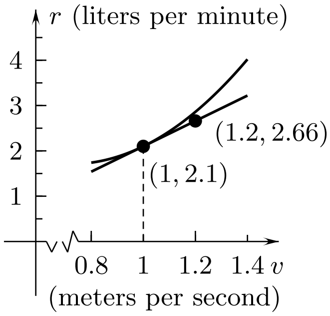
26.[A] $g'(1)$ for $g(x) = 3/(x+2)$

27.[O] $W'(x)$ for $W(x) = x/(x+10)$

28.[A] $Z'(x)$ for $Z(x) = 6 - 8/x$

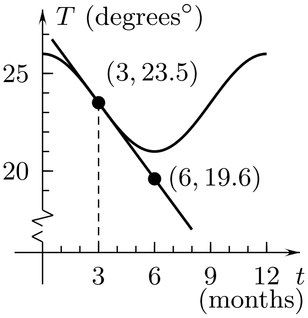
29. $y'(x)$ for $y(x) = 8 - 5x$

30.[O] Figure 29 shows the graph of the rate of oxygen consumption $r = r(v)$ (liters per minute) of a swimmer as a function of his velocity v (meters per second) and the tangent line to the graph at $v = 1$.[2] **(a)** Why is $r(1.4)$ greater than $r(1)$? **(b)** What is the rate of change of his oxygen consumption with respect to velocity at $v = 1$? (Give the units.)



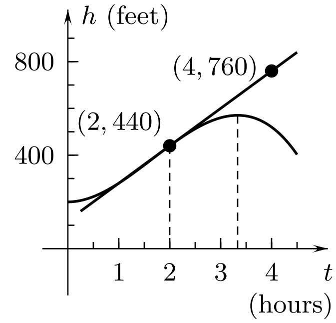
$r = r(v)$

FIGURE 29



$T = T(t)$

FIGURE 30



$h = h(t)$

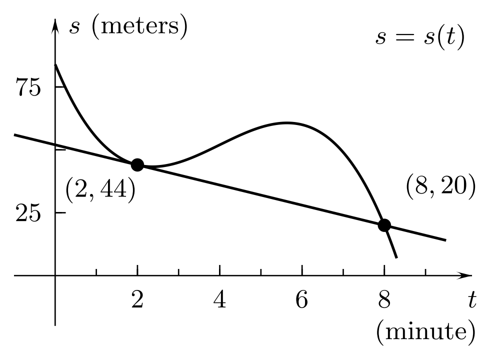
FIGURE 31

31.[A] The temperature of the ocean $T = T(t)$ (degrees Celsius) at 30° south in the southern hemisphere at time t (months) with $t = 0$ on January 1 is given by the function of Figure 30, which also shows the tangent line at $t = 3$.[3] **(a)** Why is plausible that $T(6)$ less than $T(0)$? **(b)** What is the rate of change of the temperature with respect to time at $t = 3$? (Give the units.)

32. Figure 31 shows the graph of the height $h = h(t)$ (feet) of a balloon as a function of the time t (hours) and the tangent line to the graph at $t = 2$. **(a)** During approximately what portions of the time interval $0 \le t \le 4.5$ is the balloon rising? **(b)** What is the upward velocity of the balloon at $t = 2$?

33. The function $s = s(t)$ of Figure 32 gives the s-coordinate of an object on an s-axis as a function of the time. The drawing also shows the tangent line to the graph at $t = 2$, the point $(2, 44)$ of tangency, and the point $(8, 20)$ where the curve and tangent line also intersect. **(a)** What is the instantaneous rate of change of s with respect to t at $t = 2$? **(b)** What is the average rate of change of s with respect to t for $2 \le t \le 8$?

FIGURE 32



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[2] Data adapted from *The Human Machine* by R. Alexander, New York, NY: Columbia University Press, 1992, p. 117.

[3] Data adapted from "Monsoons" by P. Webster, *Scientific American*, August, 1981, New York, NY: Scientific American, Inc, 1981, p. 110.

EXPLORATION:

Use definition **(3)** or **(7)** to find the derivatives in Exercises 34 through 39. The letters a, b, c, and k are constant parameters.

34.[O] $f'(1)$ for $f(x) = ax^2 + b$

35.[O] $g'(3)$ for $g(x) = 1/(x + k)$

36.[A] $h'(0)$ for $h(x) = ax^5 + bx^3 + cx$

37.[O] $y'(x)$ for $y = a + b/x$

38.[A] $z'(x)$ for $z = (x + k)^2$

39. $w'(x)$ for $w = ax^2 + bx + c$

Use definition **(3)** or **(7)** and rationalization of differences of square roots as necessary to find the derivatives in Exercises 40 through 45.

40.[O] $h'(6)$ for $h(x) = \sqrt{10 - x}$

41.[A] $y'(1)$ for $y = x^{1/2} - x^{-1/2}$

42. $f'(9)$ for $h(x) = 1/\sqrt{x}$

43.[A] $g'(2)$ for $g(x) = \sqrt{x^2 + 5}$

44.[O] $f'(x)$ for $f(x) = \sqrt{x}$

45. $g'(x)$ for $f(x) = \sqrt{4 - x}$

46.[A] A walker has gone $s(t)$ miles past Reklaw, Texas, t hours after dawn one summer day. What do each of the following values tell about the hike: **(a)** $s(3) = 7$, **(b)** $s'(3) = 3$, **(c)** $s''(5) = -0.5$, and **(d)** $\dfrac{s(5) - s(0)}{5} = 2$?

47.[A] The median price $N = N(t)$ (thousand dollars) of new houses and $E = E(t)$ (thousand dollars) of existing houses on the market in the U.S. in year t AD satisfy **(a)** $N(1991) = N(1970) + 97$, **(b)** $N(1991) = 1.25E(1991)$, **(c)** $E(1991) = 5E(1970)$, **(d)** $E'(1991) > 0$, and **(e)** $N'(1991) = -0.9E'(1991)$.[4] Rephrase each of these equations as an English sentence without using the symbols N and E for the functions.

48. The amount of tar $T = T(t)$ (milligrams) and the amount of nicotine $N = N(t)$ (milligrams) in an average cigarette that was manufactured in the United States in year t AD has the properties[5] **(a)** $T(1957) = 38$, **(b)** $T'(1957) = -6$, **(c)** $N(1980) - N(1957) = 24$, and **(d)** $N'(1980) = \frac{1}{3}N'(1957)$. Rephrase each of these equations as an English sentence without using the symbols T and N for the functions.

49. A small airplane whose engine is running at 2500 revolutions per minute has air speed $v = v(h)$ nautical miles per hour (knots) and uses $g = g(h)$ gallons of gasoline per hour when it is at an altitude of h thousand feet above the ground.[6] Express the following statements with equations involving $v(h), g(h)$, and their derivatives, under the assumption that the plane's engine is running at 2500 revolutions per minute. **(a[A])** The plane's air speed is 116 nautical miles per hour when it is at an altitude of 4000 feet. **(b[A])** The plane's air speed is decreasing at the rate of 0.5 knots per thousand feet and its rate of gasoline consumption is decreasing 0.2 gallons per hour per thousand feet when the plane is at an altitude of 6000 feet. **(c)** The rate at which the plane uses gasoline at an altitude of 12,000 feet is 81% of the rate at 4000 feet.

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[4] Data adapted from *1993 Statistical Abstract of the United States*, Washington, DC: U.S. Department of Commerce, 1993, p. 710.

[5] Data adapted from *Drugs Society and Human Behavior*, by O. Ray and C. Ksir, Saint Louis, MO: Mosby College Publishing, 1990, p.198.

[6] Data adapted from *Cessna Skyhawk Information Manual*, Wichita, KA: Cessna Aircraft Company, 1978, pp. 5-16.

50. In a study of vehicle sales in the United States, passenger vehicles were classified into two groups, trucks (including pickups, mini-vans, and sport-utility vehicles) and cars.[7] Express the following three statements as equations involving values of the percent $T = T(t)$ of passenger vehicles sold in year t AD that were trucks, the percent $C = C(t)$ that were cars, and their derivatives: **(a)** Twenty percent of the passenger vehicles that were sold were trucks in 1970. **(b)** Thirty-eight percent were trucks in 1992. **(c)** The percent of passenger vehicles sold that were trucks was increasing 0.4 percent per year in 1992. **(d)** What are the values of $C(1970), C(1992)$, and $C'(1992)$?

51.[A] The derivative of $F(x) = x^4$ at $x = 1$ is $F'(1) = 4$. Find, by trial and error, a difference quotient $(b^4 - 1)/(b - 1)$ that approximates the derivative with an error that is less than 0.01 and greater than 0.005.

52. The derivative of $P(x) = \sqrt{x}$ at $x = 4$ is $P'(4) = \frac{1}{4}$. Find, by trial and error, a difference quotient $(\sqrt{b} - 2)(b - 4)$ that approximates the derivative with an error that is less than 0.0005 and greater than 0.0001.

53.[A] Figure 33 shows the graph of the volume $V = V(t)$ of air in a balloon as a function of the time t with the point of tangency $(5, 40)$ and a second point $(8, k)$ on the tangent line to the graph at $t = 5$. What is the value of the constant k if $V'(5) = 15$ cubic feet per hour?

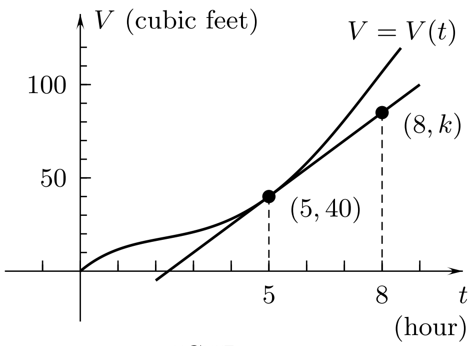

FIGURE 33

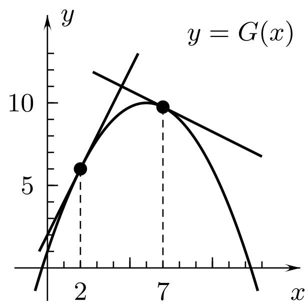

FIGURE 34

54. The tangent lines to the graph $y = G(x)$ at $x = 2$ and $x = 7$ in Figure 34 are perpendicular. Also $G'(2) = 2$. What is the value of the derivative $G'(7)$ at $x = 7$?

55. Use Definition 1 to find the derivative of $y = x^{1/3}$ at $x = 1$. ("Rationalize" the difference quotient by multiplying the numerator and denominator of the difference quotient by $x^{2/3} + x^{1/3} + 1$ and then using the identity $(a - b)(a^2 + ab + b^2) = a^3 - b^3$.

56. Find the derivative of $y = \sqrt{x}$ at $x = 5$ as the limit of $\dfrac{y - \sqrt{5}}{y^2 - 5}$ as $y \to \sqrt{5}$.

(End of Section 2.3)

[7] Data adapted from *The New York Times*, August 4, 1993, New York, NY: The New York Times Company, p. C1. Source: Ford Motor Company.