

Derivatives of power functions and linear combinations

OVERVIEW: In this section we derive rules for differentiating power functions $y = x^n$ and linear combinations $y = Af(x) + Bg(x)$ of functions whose derivatives are known.

Topics:

- Leibniz notation and the differentiation operator
- The derivative of $y = x^n$
- Derivatives of linear combinations of functions

Leibniz notation and the differentiation operator

Isaac Newton (1642–1727) and Gottfried Leibniz (1646–1716), who are considered to be the founders of calculus, each introduced notation for the derivative. Newton employed notation similar to the “prime notation” $f'(a)$ that we used in the last section. The symbols used by Leibniz evolved into what is known today as LEIBNIZ NOTATION. With this notation, the derivative f' of f is denoted $\frac{df}{dx}$, its value at $x = a$

is denoted $\left. \frac{df}{dx} \right|_{x=a}$, and the Δx -formulation of the definition of the derivative from Section 2.3 becomes

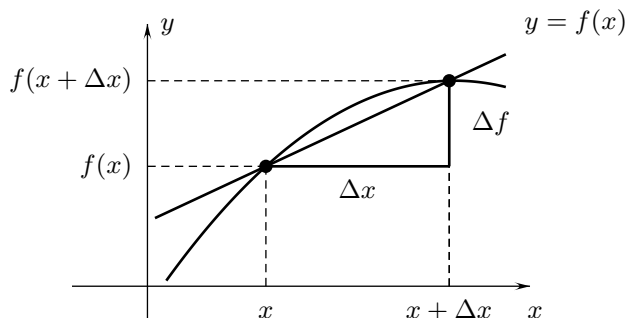
$$\left. \frac{df}{dx} \right|_x = f'(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}. \quad (1)$$

As we saw in the last section, Δx is the run from $(x, f(x))$ to $(x + \Delta x, f(x + \Delta x))$ on the secant line, as shown in Figure 1, and Δf is the corresponding rise $f(x + \Delta x) - f(x)$.

Secant line of slope

$$\frac{\Delta f}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

FIGURE 1



One advantage of Leibniz notation is that it includes the name of the variable with respect to which the derivative is being taken. It is also used in the symbol d/dx for the DIFFERENTIATION OPERATOR that transforms a function f into its derivative df/dx . We write

$$\frac{d}{dx}[f(x)] = \left. \frac{df}{dx} \right|_x$$

or with prime notation for the derivative

$$\frac{d}{dx}[f(x)] = f'(x).$$

The differentiation operator d/dx provides a convenient way to express derivatives of functions given by formulas. We will use it in the following discussion of the derivative $\frac{d}{dx}(1)$ of the function $y = 1$, the derivative $\frac{d}{dx}(x)$ of the function $y = x$, and other cases of $\frac{d}{dx}(x^n)$ with constant n .

The derivatives of $y = 1$ and $y = x$

Because the constant function $y = 1$ and the function $y = x$ are linear, their derivatives are the slopes of the lines that are their graphs (Figures 2 and 3):

$$\frac{d}{dx}(1) = 0 \quad \text{and} \quad \frac{d}{dx}(x) = 1. \quad (2)$$

We can also obtain these formulas from formula (2) for the derivatives, as in the following example.

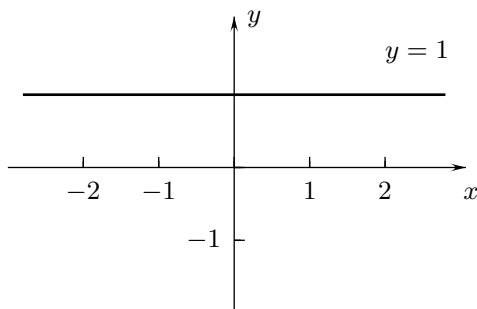
Example 1 Derive formulas (2) from definition (1).

SOLUTION If $f(x) = 1$, then $f(x + \Delta x) = 1$ for any Δx and by (1),

$$\frac{d}{dx}(1) = \lim_{\Delta x \rightarrow 0} \frac{1 - 1}{\Delta x} = \lim_{\Delta x \rightarrow 0} (0) = 0.$$

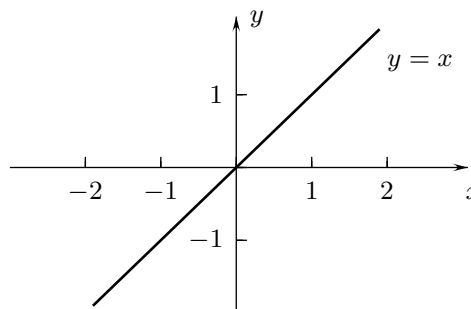
For $f(x) = x$, we have $f(x + \Delta x) = x + \Delta x$

$$\frac{d}{dx}(x) = \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x) - x}{\Delta x} = \lim_{\Delta x \rightarrow 0} (1) = 1. \quad \square$$



$$\frac{d}{dx}(1) = [\text{Slope}] = 0$$

FIGURE 2



$$\frac{d}{dx}(x) = [\text{Slope}] = 1$$

FIGURE 3

The derivative of $y = x^n$

Formulas (2) are special cases of the following general result.

Theorem 1 For any constant n and for x in the open interval(s) where $y = x^{n-1}$ is defined,

$$\frac{d}{dx}(x^n) = nx^{n-1} \quad (3a)$$

or with prime notation

$$(x^n)' = nx^{n-1}. \quad (3b)$$

The derivative (3) exists for all x if $n = 0$ or $n = 1$, as we saw in Example 1. It does not exist for $x \leq 0$ if n is irrational or if n is a fraction p/q with q an even integer, since then the domain of $y = x^n$ does not include any negative numbers. It does not exist at $x = 0$ if n is < 1 since then $y = x^{n-1}$ is not defined at $x = 0$.

Proof of Theorem 1 for integers $n \geq 2$:[†] Consider a positive x and an arbitrary integer $n \geq 2$. By definition (1),

$$\frac{d}{dx}(x^n) = \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^n - x^n}{\Delta x}. \quad (4)$$

The BINOMIAL THEOREM gives

$$\begin{aligned} (x + \Delta x)^2 &= x^2 + 2x\Delta x + (\Delta x)^2 \\ (x + \Delta x)^3 &= x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 \\ (x + \Delta x)^4 &= x^4 + 4x^3\Delta x + 6x^2(\Delta x)^2 + 4x(\Delta x)^3 + (\Delta x)^3 \end{aligned}$$

in the cases of $n = 2, 3$, and 4. These and similar formulas for larger values of n imply that

$$(x + \Delta x)^n = x^n + nx^{n-1}\Delta x + [\text{Terms involving } (\Delta x)^2 \text{ and higher powers of } \Delta x].$$

This formula then yields

$$(x + \Delta x)^n - x^n = nx^{n-1}\Delta x + [\text{Terms involving } (\Delta x)^2 \text{ and higher powers of } \Delta x]$$

so that

$$\frac{(x + \Delta x)^n - x^n}{\Delta x} = nx^{n-1} + [\text{Terms involving } \Delta x \text{ and higher powers of } \Delta x].$$

The terms involving Δx and higher powers of Δx tend to zero as Δx tends to zero, and by definition (4),

$$\frac{d}{dx}(x^n) = \lim_{\Delta x \rightarrow 0} \{nx^{n-1} + [\text{Terms involving } \Delta x \text{ and higher powers of } \Delta x]\} = nx^{n-1}.$$

This is formula (3a) for integers $n \geq 2$. **QED**

Proof of Theorem 1 for positive rational numbers n : Suppose that n is a fraction $n = p/q$ with positive integers p and q and x is a positive number. We define

$$z = x^{1/q} \quad \text{and} \quad \Delta z = (x + \Delta x)^{1/q} - x^{1/q}$$

for small nonzero Δx (see Figure 4), so that $(x + \Delta x)^{1/q} = z + \Delta z$. Then

$$x^{p/q} = z^p, \quad (x + \Delta x)^{p/q} = (z + \Delta z)^p, \quad \text{and} \quad \Delta x = (z + \Delta z)^q - z^q. \quad (5)$$

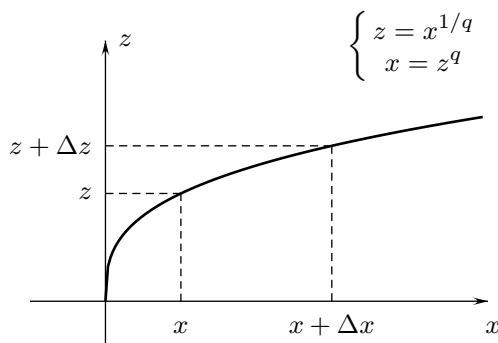


FIGURE 4

[†]A derivation of (3) for integers $n \geq 2$ using factoring is outlined in Problems 1 through 3 of this section, and in Section 2.7 we will give another proof using the Product Rule and Mathematical Induction.

Since $\Delta z \rightarrow 0$ as $\Delta x \rightarrow 0$, definition (4) with $n = p/q$ and formulas (5) yield

$$\begin{aligned} \frac{d}{dx}(x^{p/q}) &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^{p/q} - x^{p/q}}{\Delta x} = \lim_{\Delta z \rightarrow 0} \frac{(z + \Delta z)^p - z^p}{(z + \Delta z)^q - z^q} \\ &= \frac{\lim_{\Delta z \rightarrow 0} \frac{(z + \Delta z)^p - z^p}{\Delta z}}{\lim_{\Delta z \rightarrow 0} \frac{(z + \Delta z)^q - z^q}{\Delta z}}. \end{aligned}$$

At the last step we divided the numerator and denominator by Δz and expressed the limit of the quotient as the quotient of the limits. The numerator and denominator of the last fraction are, by definition, the z -derivatives of $y = z^p$ and $y = z^q$, so we obtain from Theorem 1 for positive integers n

$$\begin{aligned} \frac{d}{dx}(x^{p/q}) &= \frac{\frac{d}{dz}(z^p)}{\frac{d}{dz}(z^q)} = \frac{pz^{p-1}}{qz^{q-1}} = \left(\frac{p}{q}\right)z^{p-q} \\ &= \left(\frac{p}{q}\right)(x^{1/q})^{p-q} = \left(\frac{p}{q}\right)x^{p/q-1}. \end{aligned}$$

This gives (3a) for the arbitrary positive fraction $n = p/q$. **QED**

Proof of Theorem 1 for negative rational numbers n : To treat the case of a negative integer or fraction n , we set $n = -m$. Then m is a positive integer or fraction for which we have established (4). We write

$$\begin{aligned} \frac{d}{dx}(x^{-m}) &= \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^{-m} - x^{-m}}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left(\frac{1}{(x + \Delta x)^m} - \frac{1}{x^m} \right) \\ &= \lim_{\Delta x \rightarrow 0} \frac{1}{\Delta x} \left[\frac{x^m - (x + \Delta x)^m}{x^m(x + \Delta x)^m} \right] = \lim_{\Delta x \rightarrow 0} \frac{-1}{x^m(x + \Delta x)^m} \left[\frac{(x + \Delta x)^m - x^m}{\Delta x} \right]. \end{aligned}$$

Since $(x + \Delta x)^m$ tends to x^m and the expression in the last square brackets tends to the derivative of x^m as $\Delta x \rightarrow 0$, we obtain

$$\frac{d}{dx}(x^{-m}) = \frac{-1}{x^{2m}} \frac{d}{dx}(x^m) = \frac{-mx^{m-1}}{x^{2m}} = -mx^{-m-1}.$$

This gives (4) for $n = -m$. These arguments can be modified to deal with the relevant cases of Theorem 1 with rational n and $x \leq 0$. **QED**

Theorem 1 is derived for irrational n in Section 4.2 by using logarithms.

Example 2 Find formulas for (a) $\frac{d}{dx}(x^3)$, (b) $\frac{d}{dx}(x^{-1})$, and (c) $\frac{d}{dx}(x^{1/2})$.

SOLUTION Theorem 1 with $n = 3, -1$, and $\frac{1}{2}$ gives (a) $\frac{d}{dx}(x^3) = 3x^{3-1} = 3x^2$,

(b) $\frac{d}{dx}(x^{-1}) = (-1)x^{-1-1} = -x^{-2}$, and (c) $\frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{(1/2)-1} = \frac{1}{2}x^{-1/2}$. \square

Example 3 Give an equation for the tangent line to $y = \sqrt{x}$ at $x = 9$ and draw it with the curve.

SOLUTION By part (c) of Example 2, the derivative of $y = \sqrt{x}$ at $x = 9$ is

$$y'(9) = \left[\frac{d}{dx}(x^{1/2}) \right]_{x=9} = \left[\frac{1}{2}x^{-1/2} \right]_{x=9} = \left[\frac{1}{2\sqrt{x}} \right]_{x=9} = \frac{1}{6}.$$

The tangent line has the equation $y = y(9) + y'(9)(x - 9)$, which is $y = 3 + \frac{1}{6}(x - 9)$ since $y(9) = \sqrt{9} = 3$. The curve and tangent line are shown in Figure 5. \square

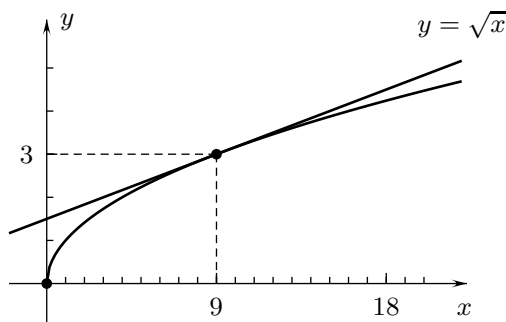


FIGURE 5

Derivatives of linear combinations of functions

A LINEAR COMBINATION of functions $y = f(x)$ and $y = g(x)$ is a function of the form $y = Af(x) + Bg(x)$ with constants A and B . Our next differentiation rule enables us to find the derivative of any such function if we know the derivatives of f and g .

Theorem 2 If $y = f(x)$ and $y = g(x)$ have derivatives at x , then for any constants A and B

$$\frac{d}{dx}[Af(x) + Bg(x)] = Af'(x) + Bg'(x). \quad (6)$$

Proof: We use definition (2) and write

$$\begin{aligned} \frac{d}{dx}[Af(x) + Bg(x)] &= \lim_{\Delta x \rightarrow 0} \frac{[Af(x + \Delta x) + Bg(x + \Delta x)] - [Af(x) + Bg(x)]}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \left\{ A \left[\frac{f(x + \Delta x) - f(x)}{\Delta x} \right] + B \left[\frac{g(x + \Delta x) - g(x)}{\Delta x} \right] \right\} \\ &= Af'(x) + Bg'(x). \quad \text{QED} \end{aligned}$$

Example 4 Find a formula for the derivative of $y = 5x^3 - x^{-2} + 4$.

SOLUTION To apply (6), we write

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(5x^3 - x^{-2} + 4) = 5 \frac{d}{dx}(x^3) - \frac{d}{dx}(x^{-2}) + 4 \frac{d}{dx}(1) \\ &= 5(3x^{3-1}) - (-2x^{-2-1}) + 4(0) = 15x^2 + 2x^{-3}. \quad \square \end{aligned}$$

Derivatives of functions involving roots of x and reciprocals of powers and roots are found by converting to exponential notation, as in the next example.

Example 5 What is $f'(x)$ for $f(x) = 7\sqrt[3]{x} + \frac{8}{\sqrt{x}}$?

SOLUTION Since $\sqrt[3]{x} = x^{1/3}$ and $\frac{1}{\sqrt{x}} = x^{-1/2}$,

$$\begin{aligned} f'(x) &= \frac{d}{dx} \left(7\sqrt[3]{x} + \frac{8}{\sqrt{x}} \right) = 7\frac{d}{dx}(x^{1/3}) + 8\frac{d}{dx}(x^{-1/2}) \\ &= 7 \left(\frac{1}{3}x^{(1/3)-1} \right) + 8 \left(-\frac{1}{2}x^{-(1/2)-1} \right) = \frac{7}{3}x^{-2/3} - 4x^{-3/2}. \quad \square \end{aligned}$$

Example 6 What is the derivative of $y = \frac{ax^2 + bx + c}{\sqrt{x}}$ for constants a, b , and c ?

SOLUTION We first rewrite the formula for y as a linear combination of powers of x :

$$y = \frac{ax^2 + bx + c}{\sqrt{x}} = \frac{ax^2 + bx + c}{x^{1/2}} = ax^{3/2} + bx^{1/2} + cx^{-1/2}.$$

Then we apply Theorems 1 and 2:

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(ax^{3/2} + bx^{1/2} + cx^{-1/2}) = a\frac{d}{dx}(x^{3/2}) + b\frac{d}{dx}(x^{1/2}) + \frac{d}{dx}(x^{1/2}) \\ &= a\left(\frac{3}{2}x^{1/2}\right) + b\left(\frac{1}{2}x^{-1/2}\right) + c\left(-\frac{1}{2}x^{-3/2}\right) \\ &= \frac{3}{2}ax^{1/2} + \frac{1}{2}bx^{-1/2} - \frac{1}{2}cx^{-3/2}. \quad \square \end{aligned}$$

Example 7 A man driving on a straight road is $s = 5t^2 + 20t + 40$ miles from his home t hours after noon. What is his car's velocity at 4:00 PM?

SOLUTION The velocity at time t is $\frac{ds}{dt} = \frac{d}{dt}(5t^2 + 20t + 40) = 10t + 20$, so at $t = 4$ (4:00 PM) it is

$$\left. \frac{ds}{dt} \right|_{t=4} = \left[10t + 20 \right]_{t=4} = 10(4) + 20 = 60 \text{ miles per hour. } \square$$

Example 8 A precision heater is controlled by varying the current supplied to it. It produces $Q(I) = 100I^2$ Calories of heat in one second when the current is I amperes. **(a)** What is the (instantaneous) rate of change of Q with respect to I at $I = 3$? **(b)** Give an equation for the tangent line to the graph $Q = Q(I)$ at $I = 3$ and draw it with the curve in an IQ -plane.

SOLUTION **(a)** The instantaneous rate of change of Q with respect to I is $Q'(I) = \frac{d}{dI}(100I^2) = 200I$, so that $Q'(3) = 200(3) = 600$ Calories per ampere. **(b)** Since $Q(3) = 100(3^2) = 900$, the tangent line is $Q = 900 + 600(I - 3)$. It is shown with the curve in Figure 6. \square

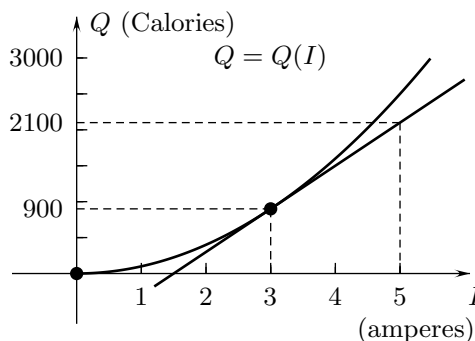


FIGURE 6

Interactive Examples 2.4

Interactive solutions are on the web page <http://www.math.ucsd.edu/~ashenk/>.[†]

- Find the derivative $\frac{d}{dt}(4t^{-1/3} - 3t^{1/4})$.
- What is the t -derivative of $(\sqrt[3]{t})^2 + (\sqrt{t})^3$?
- A ball rolling straight down a hill is $s = 1 + \frac{1}{2}t^2$ feet from a tree t seconds after it is released. What are **(a)** its average velocity for $0 \leq t \leq 2$ and **(b)** its instantaneous velocity at $t = 1$? **(c)** Draw the graph of $s = s(t)$ and the secant and tangent lines whose slopes are the average and instantaneous velocities from parts (a) and (b).
- (a)** Find an equation of the tangent line to $y = 53 + 7x^{10}$ at $x = 1$. **(b)** Generate the curve and the tangent line together in the window $-0.5 \leq x \leq 2, -10 \leq y \leq 130$ on a calculator or computer and copy them on your paper.

[†]In the published text the interactive solutions of these examples will be on an accompanying CD disk which can be run by any computer browser without using an internet connection.

Exercises 2.4

^AAnswer provided. ^OOutline of solution provided. ^CGraphing calculator or computer required.

CONCEPTS:

1. The formula $\left. \frac{d}{dx}(x^n) \right|_{x=a} = na^{n-1}$ from Theorem 1 can also be proved for positive integers n by factoring with the definition

$$\left. \frac{d}{dx}(x^n) \right|_{x=a} = \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a}. \quad (7)$$

Carry out this calculation for $n = 2$ to show that $\left. \frac{d}{dx}(x^2) \right|_{x=a} = 2a$.

2. Because the polynomial $x^3 - a^3$ is zero at $x = a$, its factorization can be found by dividing $x - a$ into $x^3 - a^3$ as shown below. We obtain $x^3 - a^3 = (x - a)(x^2 + ax + a^2)$. Use this with definition (7) to show that $\left. \frac{d}{dx}(x^3) \right|_{x=a} = 3a^2$.

$$\begin{array}{r} x - a \overline{) x^3 - ax^2} \\ \underline{x^3 - ax^2} \\ ax^2 \\ \underline{ax^2 - a^2x} \\ a^2x - a^3 \\ \underline{a^2x - a^3} \\ 0 \end{array}$$

3. (a) Complete the division of $x - a$ into $x^4 - a^4$ that has been started below, and use the result to show that $x^4 - a^4 = (x - a)(x^3 + ax^2 + a^2x + a^3)$. (b) Use this factorization with definition (7) to prove that $\left. \frac{d}{dx}(x^4) \right|_{x=a} = 4a^3$.

$$\begin{array}{r} x - a \overline{) x^4 - ax^3} \\ \underline{x^4 - ax^3} \\ ax^3 \\ \underline{ax^3 - a^2x^2} \\ a^3x \\ \underline{a^3x - a^4} \\ 0 \end{array}$$

4. The factorizations in Problems 1 through 3 are special cases of the formula $x^n - a^n = (x - a)(x^{n-1} + ax^{n-2} + \cdots + a^{n-2}x + a^{n-1})$ for arbitrary positive integers n , where there are n terms inside the last parentheses. Use this with definition (7) to prove that $\left. \frac{d}{dx}(x^n) \right|_{x=a} = na^{n-1}$ for such n

BASICS:

Find the derivatives in Exercises 5 through 7.

$$5.^0 \quad \frac{d}{dx}(5x^3 - 6x^2 + 7x)$$

$$6.^0 \quad \frac{d}{dx}(6x^{-3} - 4x^{-1})$$

$$7. \quad \frac{d}{dt}(t^{3/2} - t^{2/3})$$

$$8.^0 \quad \text{At what values of } x \text{ does } f'(x) \text{ equal 12 if } f(x) = x^3?$$

$$9.^0 \quad \text{(a) Give an equation of the tangent line to } y = \sqrt[3]{x} \text{ at } x = 27. \text{ (c) Generate the curve and tangent line together in the window } -20 \leq x \leq 70, -3 \leq y \leq 5 \text{ as a partial check of your answer.}$$

Find the derivatives in Exercises 10 through 26.

$$10.^A \quad \frac{d}{dx}(x^7 + 3x^2 - 15)$$

$$11.^A \quad \frac{d}{dx}(7x^{-8} - 8x^{-7} + 3)$$

$$12.^A \quad \frac{d}{dt}(3t^{1/3} + 5t^{7/5})$$

$$13.^A \quad \frac{d}{dx}(3x^4 - 6x^2 + 2)$$

$$14.^A \quad \frac{d}{dx}\left(\frac{1}{4}x^4 - 4x^{-1/4}\right)$$

$$15.^0 \quad \frac{d}{dt}\left[\left(\sqrt[2]{t}\right)^3 + \left(\sqrt{tx}\right)^{-1}\right]$$

$$16.^0 \quad \left.\frac{dy}{dx}\right|_{x=2} \text{ for } y = \frac{3}{x} - \frac{4}{x^2}$$

$$25.^0 \quad f'(10) \text{ where } f(x) = 3g(x) - 4h(x), g'(10) = 7, \text{ and } h'(10) = -3$$

$$26. \quad \left[\frac{d}{dt}(5V(t) + 6W(t) - 7Z(t))\right]_{t=0} \text{ where } V'(0) = -1, W'(0) = 4, \text{ and } Z'(0) = 3$$

$$27. \quad \text{A car travels } s = 37t \text{ miles in } t \text{ hours } (0 \leq t \leq 10). \text{ (a) Find its average velocity for } 0 \leq t \leq 10 \text{ and its instantaneous velocity at } t = 5. \text{ (b) How are the results of part (a) related and why?}$$

In Exercises 28 through 32 (a) find exact equations of the tangent lines at the given values of x .

(b) Generate the curves and the tangent lines together in the indicated windows, and copy the drawings on your paper.

$$28.^0 \quad \text{The tangent line to } y = 10x - x^4 \text{ at } x = 1 \quad (-1 \leq x \leq 3, -5 \leq y \leq 15)$$

$$29.^A \quad \text{The tangent line to } y = 4 - x^2 \text{ at } x = 1 \quad (-3 \leq x \leq 3, -5 \leq y \leq 8)$$

$$30. \quad \text{The tangent line to } y = x + x^2 + x^3 \text{ at } x = -2 \quad (-4 \leq x \leq 4, -20 \leq y \leq 20)$$

$$31. \quad \text{The tangent line to } y = x^3 - 4x \text{ at } x = 2 \quad (-3 \leq x \leq 4, -6 \leq y \leq 6)$$

$$32. \quad \text{The tangent line to } y = \frac{2}{\sqrt{x}} \text{ at } x = 1 \quad (-0.5 \leq x \leq 3, -0.5 \leq y \leq 5)$$

$$33. \quad \text{The tangent line to } y = x^2 + 27/x^2 \text{ at } x = 3 \quad (-5 \leq x \leq 5, -5 \leq y \leq 30)$$

$$34. \quad \text{The tangent line to } y = x^{-1/3} \text{ at } x = 1 \quad (-0.5 \leq x \leq 3.5, -0.5 \leq y \leq 3)$$

EXPLORATION:

- 35.^O** Use the definition of the derivative $y'(0) = \lim_{\Delta x \rightarrow 0} \frac{y(0 + \Delta x) - y(0)}{\Delta x}$ with $y(x) = x^{1/3}$ to explain why $y = x^{1/3}$ does not have a derivative at $x = 0$.
- 36.^A** Use the definition of the derivative $\frac{d}{dx}(x^2) = \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x}$ to derive the formula for the derivative of $y = x^2$.
- 37.** Use the definition of the derivative $\frac{d}{dx}(x^{-1}) = \lim_{\Delta x \rightarrow 0} \frac{(x + \Delta x)^{-1} - x^{-1}}{\Delta x}$ to derive the formula for the derivative of $y = x^{-1}$ for $x \neq 0$.
- 38.** Figure 7 shows the graphs of $y = F(x)$ and $y = G(x)$ and their tangent lines at $x = 1$. The slope of the tangent line to the graph of $y = F(x)$ is 2 and the slope of the tangent line to the graph of $y = G(x)$ is -2 . What is $H'(1)$ if $H(x) = 10F(x) + 5G(x)$?

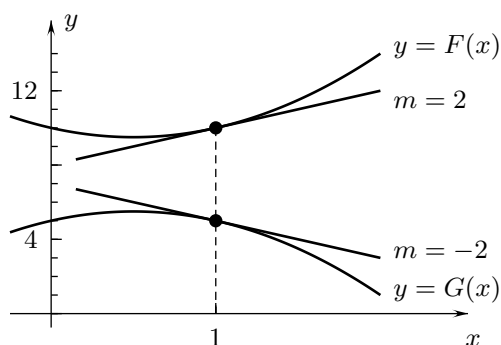


FIGURE 7

- 39.** The tangent line to the graph of $y = H(x)$ at $x = 0$ has the equation $y = 10 - x$. **(a)** Give an equation of the tangent line to the graph of $K(x) = 3H(x)$ at $x = 0$. **(b)** Give an equation of the tangent line to the graph of $L(x) = H(x) + 5$ at $x = 0$.

Find the derivatives in Exercises 40 through 47 by first rewriting the formulas so that Theorems 1 and 2 of this section can be applied.

40.^O $\frac{d}{dx} \left(\frac{\sqrt{x} - 3}{x} \right)$

41.^O $\frac{d}{dx} \sqrt{\frac{x^{3/4}}{x^{2/3}}}$

42.^A $\frac{d}{dx} [(x + 3)^2]$

43. $\frac{d}{dy} [y^2(2y^3 + 3y^2 - 4)]$

48.^A At what values of x is $w'(x) \geq 5$ if $w(x) = x^4 + x$?

- 49.^A** **(a)** The temperature in a forest is $T = 10 + 16/t^2$ degrees Fahrenheit t hours after midnight for $1 \leq t \leq 5$. Which is greater (less negative), the average rate of change of T with respect to t for $1 \leq t \leq 4$ or the instantaneous rate of change of T with respect to t at $t = 2$? **(b)** Draw the graph of $T = T(t)$ and the secant and tangent lines whose slopes are these average and instantaneous velocities.

44.^A $\frac{d}{dx} [(x + 2)(3x - 4)]$

45. $\frac{d}{dx} \left[\frac{2 + 3x^{30} - 5x^{40}}{x^{50}} \right]$

46. $\frac{d}{dx} [x^{1/7}(x^{1/6} - x^{1/5})]$

47. $\frac{d}{dt} \left[\left(\frac{t-1}{t} \right)^2 \right]$

- 50.** (a) What is the rate of change of the volume $V(w) = w^3$ of a cube with respect to the width w of its sides at $w = 3$? Give the units with w measured in yards. (b) Are the average rates of change of the volume with respect to the width for $0 \leq w \leq 3$ and for $1 \leq w \leq 3$ less than or greater than the instantaneous rate of change from part (a)? (c) Illustrate the result of part (b) by drawing the curve $V = w^3$ with the relevant tangent line and two secant lines.
- 51.** If there is no air resistance, a ball that is dropped from a height of 490 meters is $s = 490 - 4.9t^2$ meters above the ground t seconds later, until it hits the ground. (a) When does it hit the ground? (b) Draw the graph of the ball's height above the ground as a function of t . (c) What is the ball's average upward velocity as it falls? (d) What is its average downward velocity as it falls? (e) Is it falling faster three seconds or eight seconds after it is dropped?
- 52.** During a compression stroke in a diesel engine, the pressure p and volume of air V in a cylinder are related by $p = 4000V^{-7/5}$, where the pressure is measured in pounds per square inch and the volume in cubic inches.⁽¹⁾ (a) What is the rate of change of the pressure with respect to the volume when the volume is 50 cubic inches? (b) What does the fact that dp/dV is negative indicate about the engine?
- 53.** Two positive charges, of one electrostatic unit each, exert a repulsive force of $F = 1/r^2$ dynes on each other when they are r centimeters apart. (a) What is the average rate of change of the force with respect to the distance for $5 \leq r \leq 100$? (b) What is the instantaneous rate of change of the force with respect to distance when the charges are 10 centimeters apart? (c) How far apart are the charges when dF/dr equals -0.002 dynes per centimeter?
- 54.** A car is $s = 60 + 120t - 80t^{3/2}$ miles east of a town t hours after noon. (a) How fast is it traveling and in what direction is it traveling at 12:15? (b) When is its velocity zero and how far is it from the town at that time?

(End of Section 2.4)

⁽¹⁾The mathematical model $p = CV^{-7/5}$ is used here with the assumption that no heat enters or leaves the cylinder as the piston is compressed. Such a process with no heat transfer is called ADIABATIC. (*Physical Chemistry* by G. Castellan, Reading, MA: Addison Wesley Publishing Company, 1971, pp. 130–132.)